Influence of numerical conditioning on the accuracy of relative orientation

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**PURPOSE**

Effects of **numerical conditioning** in the **essential estimation**
(calibrated, overconstrained, closed-form)

- analyse the **eight-point alg.** \((8\text{pt})\) forward bias
- discuss the conditioning of **five-point alg.** \((5\text{pt})\)
- validation by comprehensive performance evaluation
Why I think this might be of interest to you:

- what causes the 8pt alg. forward bias?
- comparison of known conditioning approaches (8pt alg)
- conditioning the 5pt algorithm
- performance evaluation 5pt vs 8pt vs hg in the overconstrained case
AGENDA

- Introduction (short)
- Analysis of the 8pt forward bias
- Review of the 8pt conditioning (short)
- Conditioning the 5pt algorithm
- Experimental validation
- Conclusion
**Context:**

- re-estimating **relative orientation** on the set of inliers
- we can’t solve directly for $\mathbf{R, t}$, use intermediate objects
- $\Rightarrow$ calibrated, overconstrained, closed-form $\mathbf{E, H, ...}$
THE ESSENTIAL MATRIX

The recovery approaches rely on two constraints:

- the epipolar constraint:
  \[ q_{iB}^T \cdot E \cdot q_{iA} = 0 \]

- the calibrated (5DOF) constraint:
  \[ 2 \cdot EE^T E - \text{trace}(EE^T)E = 0 \] (v1)
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A_{n \times 9} \cdot e = 0
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\[ E = a \cdot E_6 + b \cdot E_7 + c \cdot E_8 + d \cdot E_9 \]
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This is equivalent to:
\[ e^T \cdot \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix} = 0^T \] (v2)
THE 8PT-ALG FORWARD BIAS

The i-th row of the matrix $\mathbf{A}$:

$$\mathbf{A}_i = \begin{bmatrix} x_{iB} x_{iA} & x_{iB} y_{iA} & x_{iB} & y_{iB} x_{iA} & y_{iB} y_{iA} & y_{iB} & x_{iA} & y_{iA} & 1 \end{bmatrix}$$
The 8pt-alg forward bias

The i-th row of the matrix $A$:

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Compare "quadratic" (1,2,4,5) and "linear" (3,6,7,8) columns:

$$a_{i1} = \hat{a}_{i1} + \hat{x}_{iB}\Delta x_{iA} + \Delta x_{iB}\hat{x}_{iA} + \Delta x_{iB}\Delta x_{iA}$$

$$a_{i3} = \hat{a}_{i3} + \Delta x_{iB}$$
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Under default conditions ($\alpha = 45^\circ$):

$$|a_{i1} - \hat{a}_{i1}| < |a_{i3} - \hat{a}_{i3}|$$
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The deviation ratio can be determined:

$$r_{Eql} = \sqrt{E[\text{var}(a_{i1})]/E[\text{var}(a_{i3})]} = \tan(\alpha/2) \cdot \sqrt{2/3}$$
$$r_{Eql}(\alpha = 45^\circ) = 0,33$$
$$r_{Eql}(\alpha = 102^\circ) = 1$$
THE 8PT-ALG FORWARD BIAS (2)

Estimation favours solutions $E$ with large

\[
\text{conv}(E) = \left| [E_{13}, E_{23}, E_{31}, E_{32}] \right|^{-1}
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A \times e = 0
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Conditioning in relative orientation: The 8pt-alg forward bias (2) 8/18
**THE 8PT-ALG FORWARD BIAS (2)**

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For moderate rotations $\text{conv}$ attains maximum near the *forward* direction:

$a_m = \arg \max_a \text{conv}([a] \times R) \approx [0 \ 0 \ 1]^\top$

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The bias especially affects the *planar case* when the epipolar constraint is *degenerate*: $\mathcal{E}(H) = [a] \times \cdot H, \quad \forall a \in \mathbb{R}^3$,
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However, the bias also affect the usual 3D contexts, where the distance to the target is much greater than the baseline.
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However, the bias also affect the usual 3D contexts, where the distance to the target is much greater than the baseline

Here the translation errors can be approximately compensated by slight rotation deviations; small residual changes in the whole translation spectrum!
NUMERICAL CONDITIONING

Review of the 8pt conditioning approaches:

In Hartley’s normalization, we recover $E' = T_2^{-\top}ET_1^{-1}$, relating the transformed points $q'_{ik} = T_kq_{ik}$, $k = A, B$. 
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Mühlich considers an equilibrated matrix $A_{eq} = W_L \cdot A \cdot W_R$

The new system is $A_{eq} \cdot e' = 0$, where $e' = W_R^{-1} \cdot e$

The proposed $W_R$ ensures a zero-mean expected error in $e'$
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Wu et al. have *reformulated* the linear estimation problem: the new matrix has *only* linear entries, but is $4n \times (3n + 9)$

Results similar to equilibration

The procedure is much more computationally demanding
CONDITIONING THE 5PT ALGORITHM

Although the individual right-singular vectors are very sensitive, their span is quite stable!

Deviations $\delta_i = \min(|e_i - \hat{e}_i|, |e_i + \hat{e}_i|)$, sidewise motion, $N=10^4$, $\sigma=1$:

$\alpha_H=45^\circ$, 3D scene  $\alpha_H=45^\circ$, planar scene  $\alpha_H=120^\circ$, 3D scene
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Hence, the conditioning much less beneficial than with 8ptAlg.
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**CONDITIONING THE 5PT ALGORITHM**

Although the **individual** right-singular vectors are very sensitive, their **span** is quite **stable**!

Deviations $\delta_i = \min( |e_i - \hat{e}_i|, |e_i + \hat{e}_i| )$, sidewise motion, $N=10^4$, $\sigma=1$:

$\alpha_H = 45^\circ$, 3D scene       $\alpha_H = 45^\circ$, planar scene       $\alpha_H = 120^\circ$, 3D scene

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EXPERIMENTS

Parameters of the artificial experimental setup:

- geometric: $\phi, \theta, \text{distance}, \text{depth}, \text{slant}$
- imaging: $\alpha_H, \sigma, \text{resolution}^\dagger$ for $\alpha_H=45^\circ$ is $384 \times 288$

\[
\begin{align*}
\alpha_H &\quad \phi \\
\text{distance} &\quad \text{depth} \\
\text{slant} &\quad \text{resolution}^\dagger
\end{align*}
\]
EXPERIMENTS (2)

We consider the **accuracy** of the recovered epipole $t$ in variants *standard*, *hartley* and *muehlich*

We perform $10^4$ experiments with 50 random points and observe:
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- Spherical distribution of the **epipole** $t$ (the arrow denotes $\hat{t}$)
- Distribution of the **angular epipole error** $\Delta t := \angle(t, \hat{t})$

![Spherical distribution of the epipole](image1.png)

![Distribution of the angular epipole error](image2.png)
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- Spherical distribution of the epipole $t$ (the arrow denotes $\hat{t}$)
- Distribution of the angular epipole error $\Delta t := \angle(t, \hat{t})$
- Dependence of $med\{\Delta t\}$ on different parameters of the setup
**Experiments (3)**

8pt-standard epipoles in **degenerate** and **noisy** datasets:

Common: distance=10, $\alpha_H=45^\circ$
Top: depth=0, $\sigma=0$. Bottom: depth=5, $\sigma=1$. 
Left: $\theta=(120^\circ, 180^\circ)$, $\phi=0^\circ$. Right: $\theta=135^\circ$, $\phi=(-20^\circ, 20^\circ)$.

The **shifted modes** clearly reflect the forward bias.

Backward motion ($|\theta|>90^\circ$) produces $t$ with positive $z$.
EXPERIMENTS (4)

The bias goes away for large $\alpha_H$, low $\sigma$, low distance or conditioned data:

Common: $\text{distance}=10$, $\text{depth}=5$, $\theta=135^\circ$, $\phi=0^\circ$, $\alpha_H=45^\circ$, $\sigma = 1$

Top: $\alpha_H=60^\circ,90^\circ,100^\circ,120^\circ$

Bottom: $\sigma=0,2$, $\text{distance}=3$, normalization, equilibration.
Normalization and equilibration perform similarly, except for forward motion:

Common: $\text{distance}=10$, $\text{depth}=5$, $\theta=170^\circ$, $\phi=0^\circ$, $\alpha_H=45^\circ$
Left: $\sigma=0.5$, Right: $\sigma=1.0$
Top: normalization, Bottom: equilibration
EXPERIMENTS (6)

5pt vs 8pt for 3D scenes ($\text{med}\{\Delta t\}$, distance=10, depth=5)

\[ \sigma=1,0; \alpha_H=45^\circ. \]

5pt disambiguation relies on the total reprojection error.

Conditioning helps more 8pt than 5pt.
**Experiments (7)**

5pt vs hg for planar scenes ($med\{\Delta t\}$, distance=10, depth=0)

σ=1,0; $\alpha_H=45^\circ$

5pt and hg disambiguation uses groundtruth!

5pt conditioning always improves the results

hg always better than 5pt
DISCUSSION

The addressed issues:

- 8pt forward bias
- 5pt numerical conditioning
- experimental validation
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Conclusions:

- 8pt-standard performance strongly depends on $\alpha_H$
- 5pt conditioning less beneficial than 8pt conditioning
- 5pt better than 8pt for:
  - shallow scenes
  - small number of points
    - break-even point: 20 (45°), 50 (90°)
- Model selection required for best results