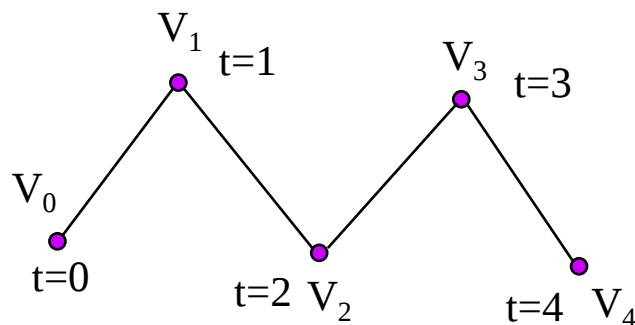


# 6. Linearna interpolacija, krivulje

## 6.1. LINEARNA INTERPOLACIJA

- parametarska jednačba pravca kroz dvije točke

$$\mathbf{V} = \mathbf{V}_0 + (\mathbf{V}_1 - \mathbf{V}_0)t = (1-t)\mathbf{V}_0 + t\mathbf{V}_1 = f_0(t)\mathbf{V}_0 + f_1(t)\mathbf{V}_1$$



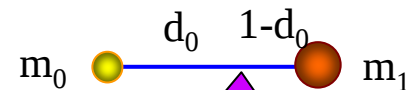
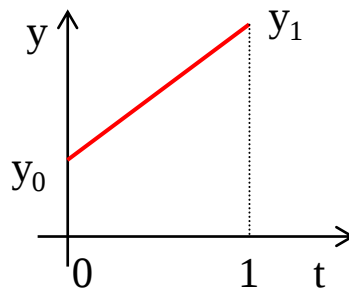
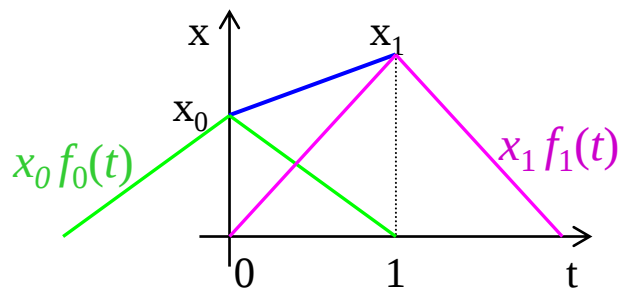
```
lerp (t, v0, v1) {  
    return v0 + t*(v1-v0);  
    // ili return (1-t)*v0 + t*v1;  
}
```

$f_0(t), f_1(t)$  težinske funkcije  
 $f_0(t) + f_1(t) = 1 \Rightarrow$  baricentrične koordinate

za zadane udaljenosti  $d_0$  i  $(1-d_0)$

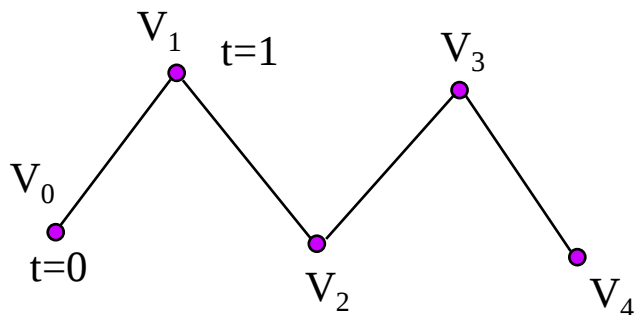
odrediti težinske funkcije  $m_i$   
 težište  $m_0 d_0 = m_1 (1-d_0)$   
 $\rightarrow m_0 = (1-d_0), m_1 = d_0$

[https://phet.colorado.edu/sims/html/balancing-act/latest/balancing-act\\_en.html](https://phet.colorado.edu/sims/html/balancing-act/latest/balancing-act_en.html)



# LINEARNA INTERPOLACIJA

- parametarska jednačba pravca kroz dvije točke

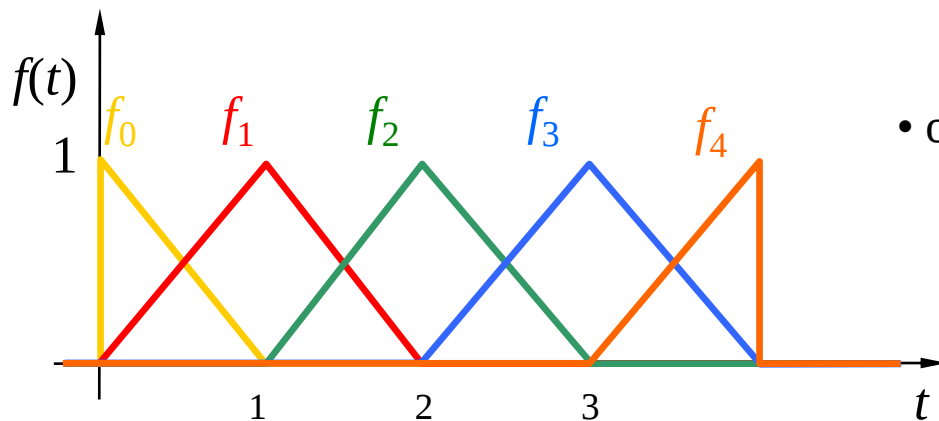


- po odsječcima linearna interpolacija

$$\mathbf{V} = \sum_i f_i(t) \mathbf{V}_i$$

$f_i(t)$  težinske funkcije

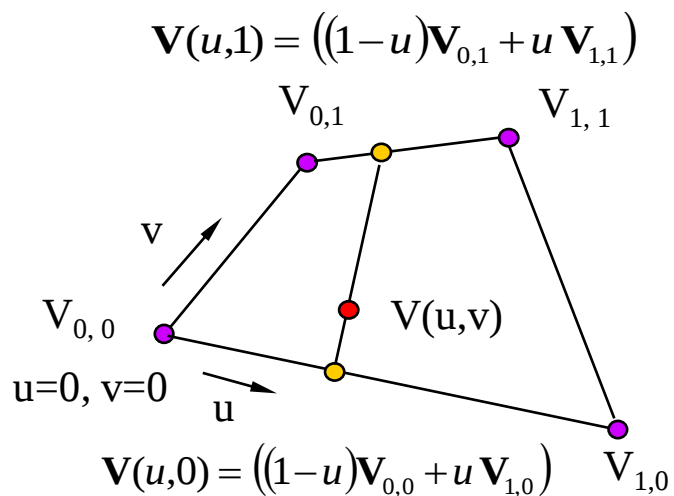
$$\sum_i f_i(t) = 1$$



- obrnuti problem od postupka uzorkovanja
  - iz niza uzoraka želimo dobiti kontinuirani signal
  - konvolucija

# BILINEARNA INTERPOLACIJA

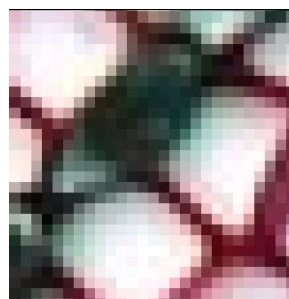
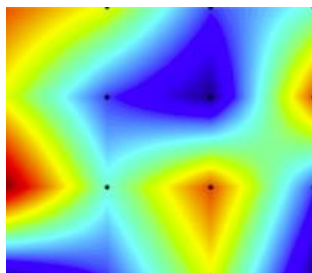
- parametarska jednažba kroz četiri točke



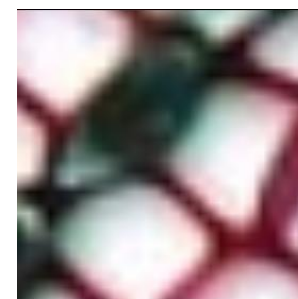
$$\mathbf{V}(u,v) = \mathbf{V}_{u,0} (1-v) + \mathbf{V}_{u,1} v$$

$$\mathbf{V}(u,v) = ((1-u)\mathbf{V}_{0,0} + u\mathbf{V}_{1,0}) (1-v) + ((1-u)\mathbf{V}_{0,1} + u\mathbf{V}_{1,1}) v$$

$$\mathbf{V}(u,v) = (1-u)(1-v)\mathbf{V}_{0,0} + u(1-v)\mathbf{V}_{1,0} + (1-u)v\mathbf{V}_{0,1} + uv\mathbf{V}_{1,1}$$



najbliži susjed



bilinearna interpolacija

<https://webgpu.github.io/webgpu-samples/samples/rotatingCube>

Primjena npr. interpolacija boje <http://micro.magnet.fsu.edu/primer/java/digitalimaging/processing/panscrollzoom/index.html>

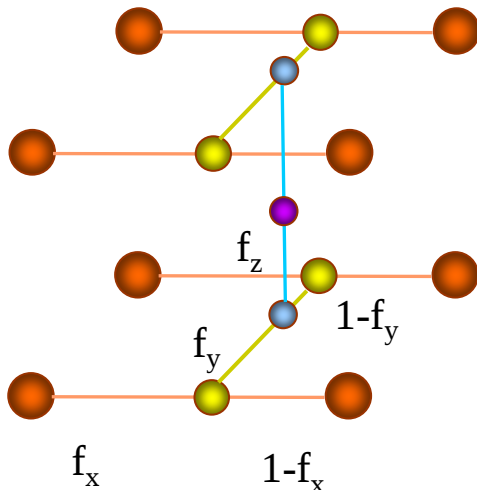
## BILINEARNA INTERPOLACIJA

- bilinearna interpolacija **nije** linearna (nije ravnina) ali je baricentrična

<http://multivis.net/lecture/bilinear.html>

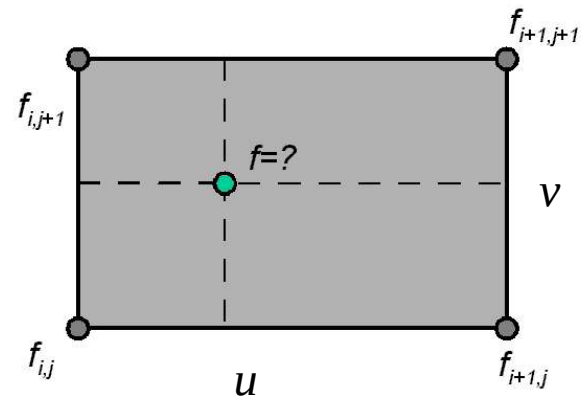
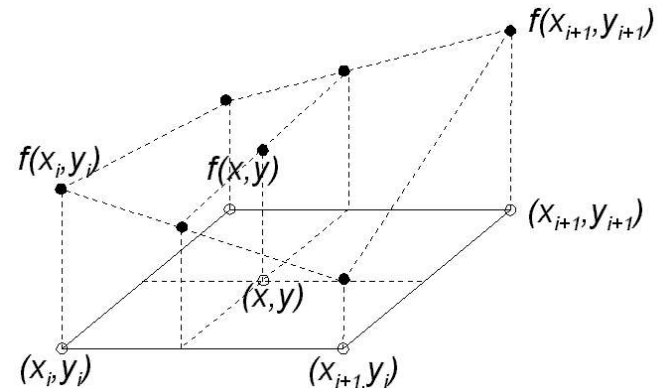
$$\mathbf{V}(u, v) = (1-u)(1-v)\mathbf{V}_{0,0} + u(1-v)\mathbf{V}_{1,0} + (1-u)v\mathbf{V}_{0,1} + uv\mathbf{V}_{1,1}$$

- trilinearna interpolacija proširenje bilinearne



<http://cs.wellesley.edu/~cs307/threejs/demos/Color/colorcube.html>

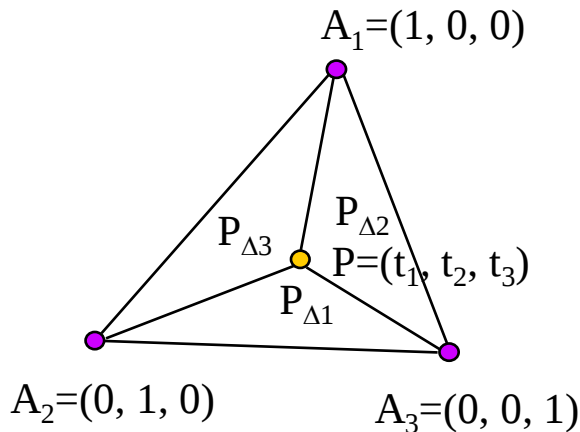
Ž. M. ZEMRIS, FER



$$u = \frac{x - x_i}{x_{i+1} - x_i}, \quad v = \frac{y - y_i}{y_{i+1} - y_i}$$

# BARICENTRIČNE KOORDINATE

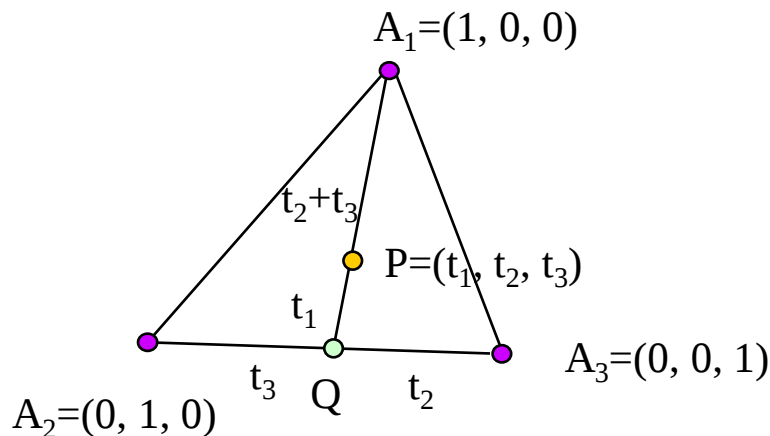
- točka P ima normalizirane baricentrične koordinate  $P(t_1, t_2, t_3)$  - Baricentar,  
 $t_1 : t_2 : t_3 = P_{\Delta 1} : P_{\Delta 2} : P_{\Delta 3}$      $t_1 + t_2 + t_3 = 1$     (ako su u  $t_i$  mase u  $A_i$ ,  $P_i$  je centar masa)



- točka je unutar trokuta ako su  $0 \leq t_1, t_2, t_3 \leq 1$
- ako je neki  $t_i = 0$  točka P pada na brid

- određivanje baricentričnih koordinata  
 kroz točke  $A_1$  i P povučemo pravac  $\rightarrow t_3, t_2$

<http://www.martin-kraus.org/LiveGraphics3D/examples/parametrized/cagd/chap3fig5.html>



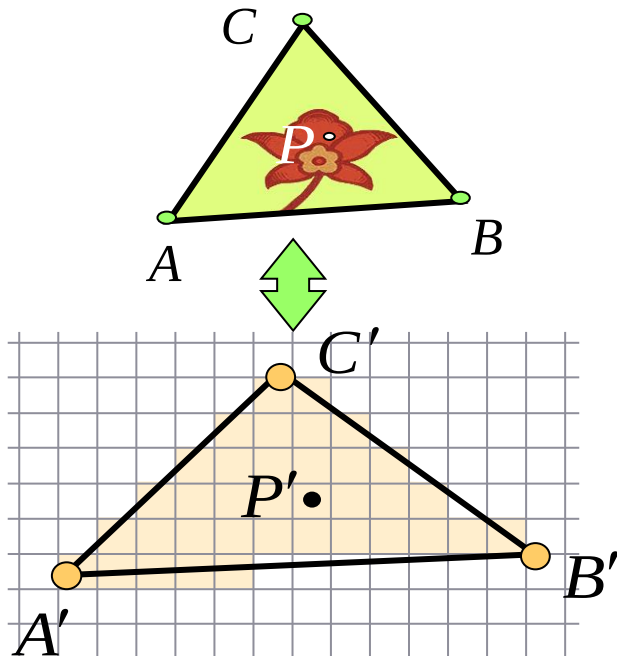
$$t_1 = \frac{P_{\Delta 1}}{P_{A_1 A_2 A_3}}$$

$$t_2 = \frac{P_{\Delta 2}}{P_{A_1 A_2 A_3}}$$

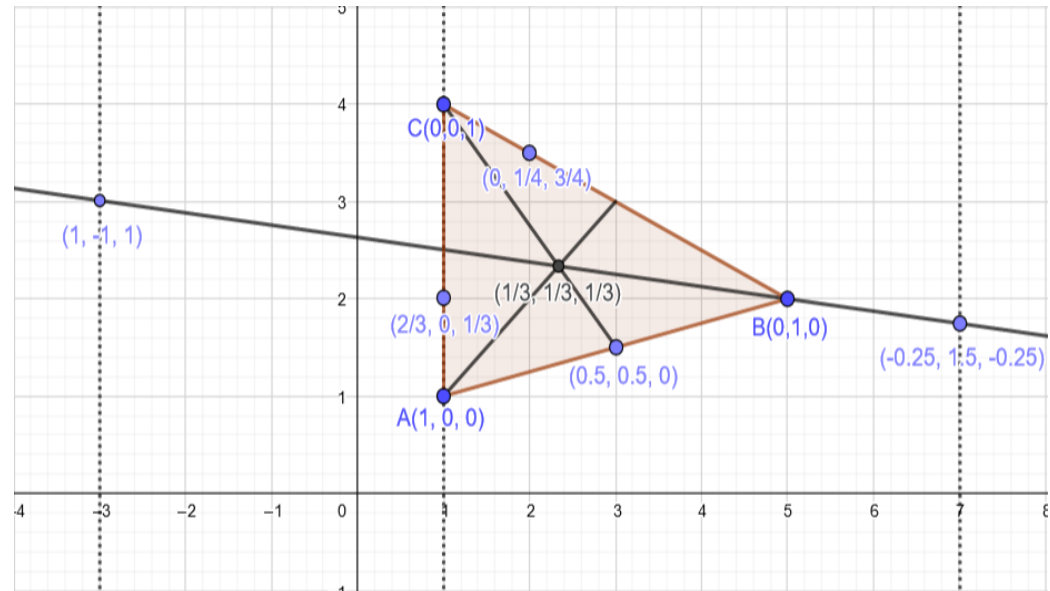
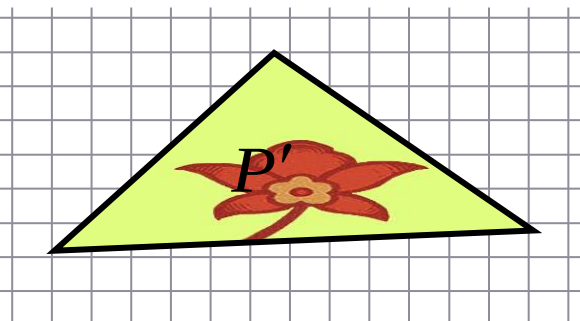
$$t_3 = \frac{P_{\Delta 3}}{P_{A_1 A_2 A_3}}$$

$$P = t_1 A_1 + t_2 A_2 + t_3 A_3$$

Primjer primjena baricentričnih koordinata: ispravno preslikavanje teksture, deformacija konveksnih objekata – preobražaj (mesh deformation, morphing)



Tagret ↓



baricentrične koordinate  $t_1, t_2, t_3$  točke  $P'$  obzirom na  $A', B', C'$  određuju boju  
 $\text{Color}(P') = \text{Color}(t_1 A + t_2 B + t_3 C)$

Npr. interpolacija boja:

(<http://www.cut-the-knot.org/Curriculum/Geometry/Barycentric.shtml>)

Proširenje na 3D: <https://lucasmajerowicz.github.io/threejs-ffd/app/>

Određivanje je li točka u trokutu:

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_0 + u(\mathbf{V}_1 - \mathbf{V}_0) + v(\mathbf{V}_2 - \mathbf{V}_0) \\ &= \mathbf{V}_0 + u \vec{\mathbf{a}} + v \vec{\mathbf{b}} \end{aligned}$$

<http://adrianboeing.blogspot.hr/2010/01/barycentric-coordinates.html>

<https://www.geogebra.org/m/c8DwbVTP>

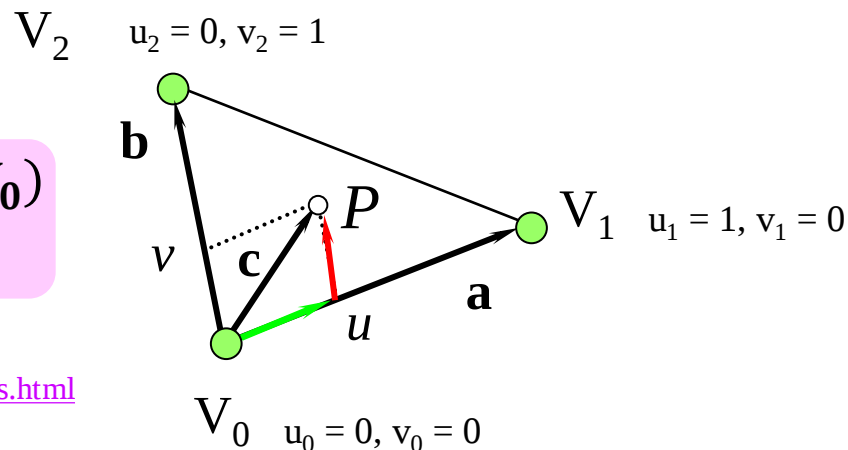
- parametarski oblik jednadžbe ravnine

$u, v$  – odgovaraju baricentričnim koordinatama  $t_1, t_2$ ,

$$u = \frac{\mathbf{b}^2 \langle \mathbf{a} \mathbf{c} \rangle - \langle \mathbf{a} \mathbf{b} \rangle \langle \mathbf{b} \mathbf{c} \rangle}{\mathbf{a}^2 \mathbf{b}^2 - \langle \mathbf{a} \mathbf{b} \rangle^2}$$

$$v = \frac{\mathbf{a}^2 \langle \mathbf{b} \mathbf{c} \rangle - \langle \mathbf{a} \mathbf{b} \rangle \langle \mathbf{a} \mathbf{c} \rangle}{\mathbf{a}^2 \mathbf{b}^2 - \langle \mathbf{a} \mathbf{b} \rangle^2}$$

$\langle \rangle$  je skalarni produkt vektora



Točka je u trokutu ako je:

$$(u > 0) \ \& \ (v > 0) \ \& \ (u + v < 1)$$

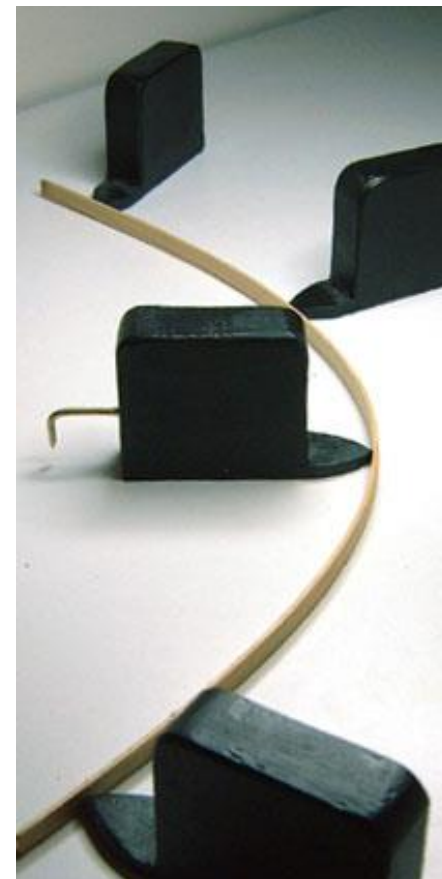
Ako je  $\mathbf{a} \perp \mathbf{b}$  možemo promatrati projekcije vektora  $\mathbf{c}$  na  $\mathbf{a}$  i vektora  $\mathbf{c}$  na  $\mathbf{b}$

$$u = \langle \mathbf{a} \mathbf{c} \rangle / |\mathbf{a}|$$

$$v = \langle \mathbf{b} \mathbf{c} \rangle / |\mathbf{b}|$$

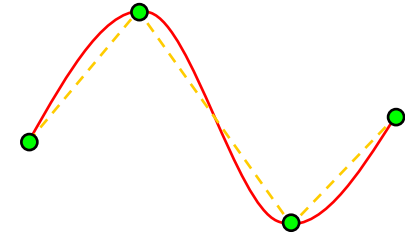
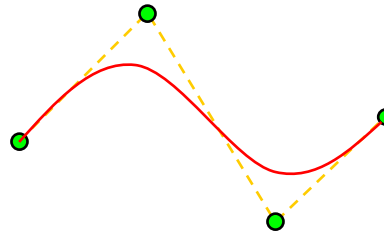
## 6.2. KRIVULJE

- postupak projektiranja krivulje
- [Animacija po krivulji](#)
  - analitički izraz izvorne krivulje u pravilu je nepoznat
  - poznato je
    - koordinate u nekim točkama
    - nagibi, zakrivljenost ili izvijanje u nekim točkama $\Rightarrow$  modeliranje
  - opis segmenta krivulje
  - segmentiranje
    - povezivanje segmenata uz ostvarivanje kontinuiteta između segmenata



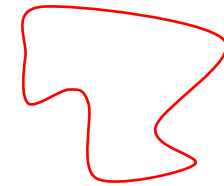
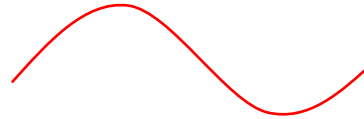
## PODJELA KRIVULJA

- aproksimacijske
- interpolacijske



[www.ibiblio.org/e-notes/Splines/lagrange.html](http://www.ibiblio.org/e-notes/Splines/lagrange.html)

- otvorene
- zatvorene



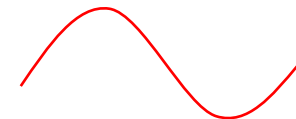
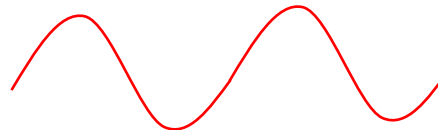
- razlomljene
- nerazlomljene

$$x(t) = \frac{a_1 t^3 + b_1 t^2 + c_1 t + d_1}{a t^3 + b t^2 + c t + d}$$

$$x(t) = a_1 t^3 + b_1 t^2 + c_1 t + d_1$$

- periodične
- neperiodične

(periodičnost težinskih funkcija)

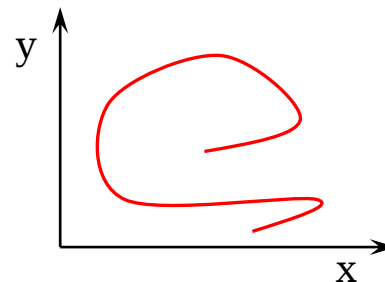


# POŽELJNA SVOJSTVA KRIVULJA

- višestruke vrijednosti

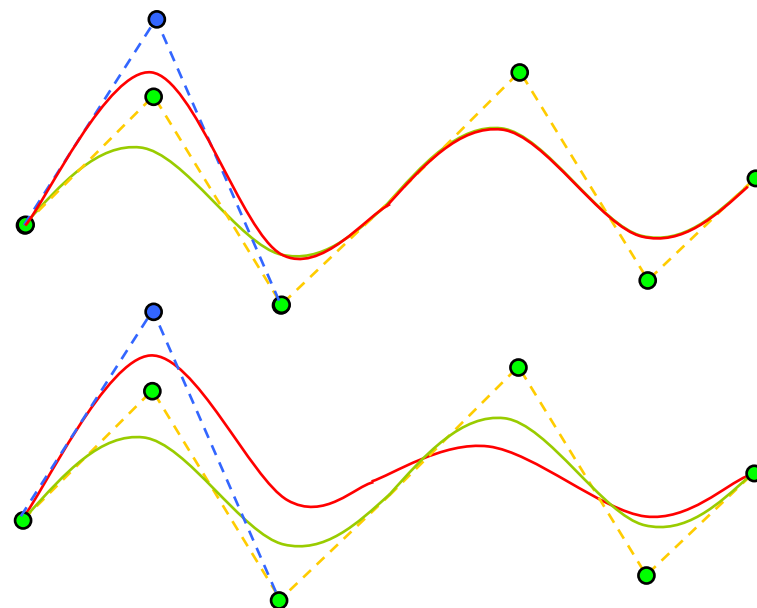
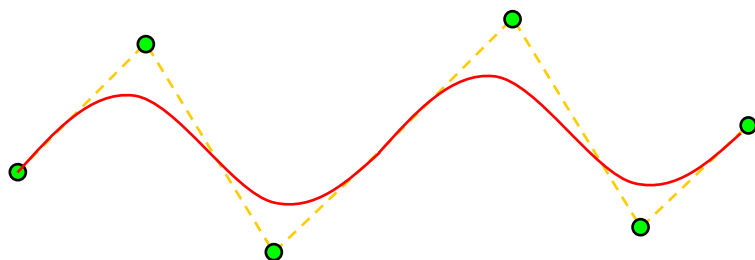
[https://mathinsight.org/parametrized\\_curve\\_arc\\_length](https://mathinsight.org/parametrized_curve_arc_length)

<https://stemkoski.github.io/Three.js/Graphulus-Curve.html>

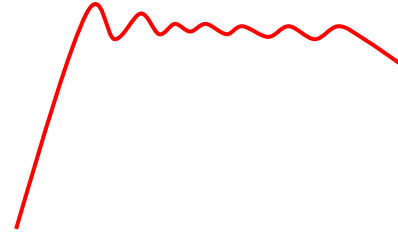


- neovisnost o koordinatnom sustavu (Kartezijev, polarni)

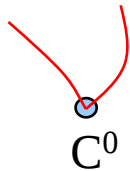
- lokalni nadzor



- smanjenje varijacije - kod visokog stupnja polinoma može se javiti titranje krivulje

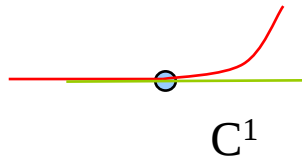


- kontrola reda neprekinutosti
- [https://mathinsight.org/applet/derivative\\_interpolating\\_polynomial\\_chebyshev\\_nodes](https://mathinsight.org/applet/derivative_interpolating_polynomial_chebyshev_nodes)



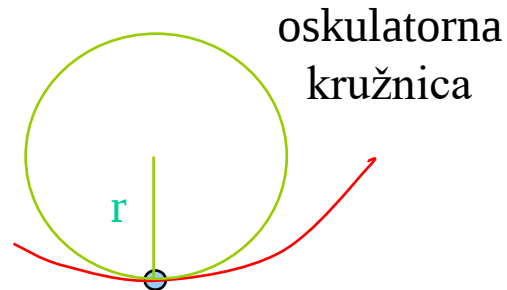
$C^0$

ista vrijednost  
koordinate



$C^1$

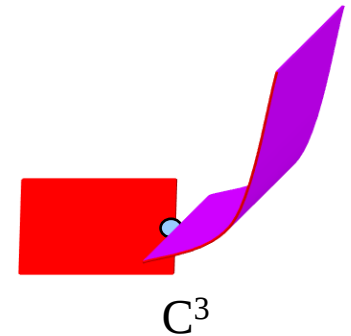
ista tangenta



$C^2$

kontinuitet radijusa  
zakrivljenosti

<http://www.martin-kraus.org/LiveGraphics3D/examples/parametrized/cagd/chap10fig4.html>



$C^3$

kontinuitet u  
izvijanju

$C^0$  – kontinuitet koordinate - ista vrijednost koordinate

$$\mathbf{f}(t) = \mathbf{g}(t)$$

$C^1$  - kontinuitet nagiba - ista vrijednost prve derivacije (vektor brzine)  $\mathbf{f}'(t) = \mathbf{g}'(t)$

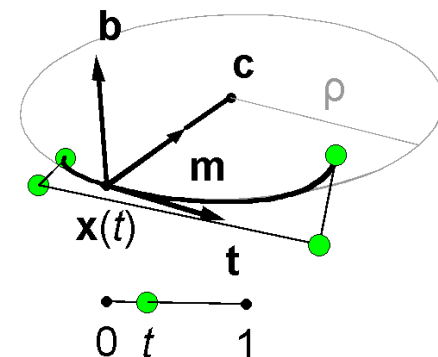
<http://math.hws.edu/graphicsbook/demos/c2/cubic-bezier.html>

$C^2$  - kontinuitet zakrivljenosti - određuje  $\mathbf{f}''(t)$  - isti vektor akceleracije

Zakrivljenost krivulje obrnuto je proporcionalna radijusu oskulatorne kružnice.

Ako je radijus velik zakrivljenost je mala (i obrnuto).

$C^3$  - kontinuitet izvijanja (torzije) - određuje  $\mathbf{f}'''(t)$



Osim  $C$  kontinuiteta postoje i  $G$  kontinuiteti koji zahtijevaju proporcionalnost.

$G$  (geometrijski)

$$G^0 = C^0$$

$G^1$  - proporcionalna vrijednost derivacije  $\mathbf{f}'(t) = k_1 \mathbf{g}'(t), k_1 > 0$

$G^2$  – ista zakrivljenost s lijeva i s desna  $\kappa = |\mathbf{f}' \times \mathbf{f}''| / |\mathbf{f}'|^3$

$G^2$  – implicira i ostvarenje svih nižih kontinuiteta  $G^1$

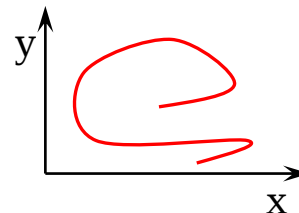
$C^1$  kontinuitet implicira  $G^1$  kontinuitet osim kada je vektor tangente  $[0 \ 0 \ 0]$

kod  $C^1$  kontinuiteta može doći do promjene smjera, kod  $G^1$  ne može.

# ANALITIČKI OPIS PROSTORNIH KRIVULJA

a) eksplicitni oblik - nemogućnost prikaza višestrukih vrijednosti

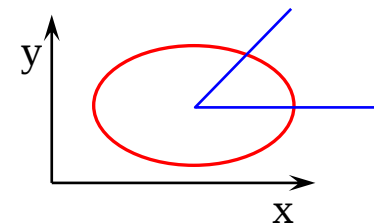
$$y = f(x), \quad z = g(x)$$



b) implicitni oblik - za prikaz dijela krivulje trebaju dodatni uvjeti

<https://stemkoski.github.io/MathBox/graph2d-implicit.html>

$$F(x, y, z) = 0$$



c) parametarski oblik

[http://mathinsight.org/applet/parametrized\\_elliptical\\_helix](http://mathinsight.org/applet/parametrized_elliptical_helix)

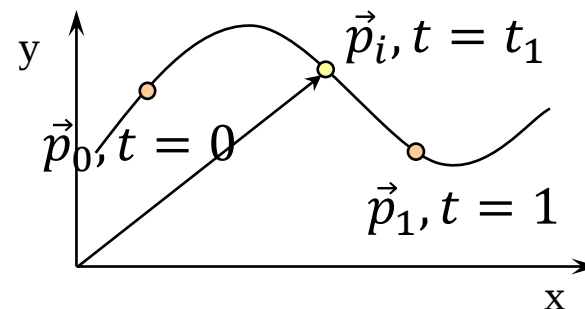
$$x = x(t), \quad y = y(t), \quad z = z(t).$$

točka na krivulji - vektorska funkcija

$$\mathbf{V}(t) = [x(t) \quad y(t) \quad z(t)]$$

vektor tangente

$$\mathbf{V}'(t) = [x'(t) \quad y'(t) \quad z'(t)]$$



$$\mathbf{V}(t_i) = \vec{p}(t_i) = \vec{p}_{ti}$$

## 6.3. SEGMENT KRIVULJE

### 6.3.1. KRIVULJA BEZIERA

Postupak poznat pod imenom krivulje Beziera nezavisno su razvili

- BEZIER 1962. Renault
- DE CASTELJAU 1959. Citroën

kao polaznu osnovu u CAD sustavima. De Casteljau direktno koristi Bernsteinove polinome.

1970. R. Forest otkriva vezu Bezierovog rada i Bernsteinovih polinoma. P. Bezier objavljuje svoj rad i krivulje dobivaju ime po njemu.

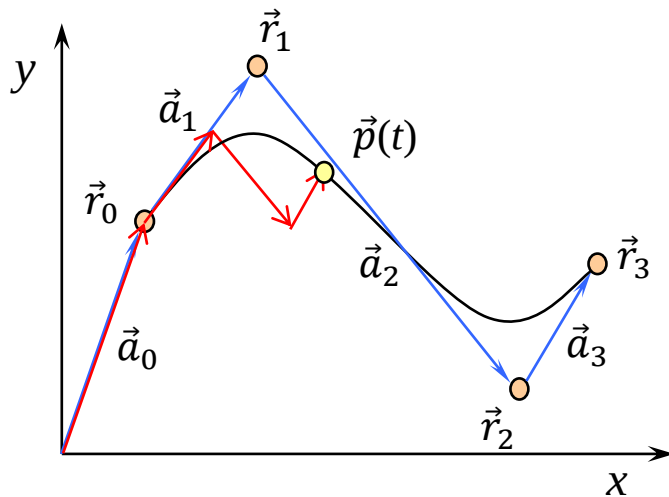
- aproksimacijske krivulje Beziera
- interpolacijske krivulje Beziera
- Bezierove težinske funkcije (Bezier)
- Bernsteinove težinske funkcije (De Casteljau)

## APROKSIMACIJSKE KRIVULJE BEZIERA

Prolaze početnom i krajnjom točkom, a ostalima se samo približava.

### a) BEZIEROVE TEŽINSKE FUNKCIJE

Korištenje gibanja vrha sastavljenog otvorenog poligona.



$$\vec{p}(t) = \sum_{i=0}^n \vec{a}_i f_{i,n}(t), t \in [0, 1]$$

$$\vec{a}_0 = \vec{r}_0$$

$$\vec{a}_i = \vec{r}_i - \vec{r}_{i-1} \quad i = 1 \dots n$$

$n+1$  .. broj točaka.

$n$  .. stupanj krivulje.

$a_i$  .. vektori kontrolnog poligona

$p(t)$  .. točka na krivulji - linearna kombinacija  $f_{i,n}(t)$  i  $a_i$ .

$f_{i,n}(t)$  .. težinska funkcija - njena vrijednost pokazuje koliko  $i$ -ti element poligona pridonosi pripadnoj točki za parametar  $t$ .

$f_{i,n}(t)$  - težinska funkcija je općenita i mora zadovoljiti niz posebnih uvjeta:

$$1. \text{ početna točka } \vec{p}(0) = \vec{a}_0 \Rightarrow f_{0,n}(0) = 1, \\ f_{i,n}(0) = 0, \quad i = 1 \dots n$$

$$2. \text{ završna točka } \vec{p}(1) = \Sigma \vec{a}_i \Rightarrow f_{i,n}(1) = 1, \quad i = 0 \dots n \quad \text{zbroy svih vektora}$$

$$3. \text{ osnovni vektor } a_1 \text{ treba biti paralelan s tangentom u početnoj točki}$$

$$\vec{p}'(0) = k_1 \vec{a}_1 \Rightarrow f'_{1,n}(0) \neq 0, \\ f'_{i,n}(0) = 0, \quad i \neq 1$$

$$4. \text{ osnovni vektor } a_n \text{ treba biti paralelan s tangentom u završnoj točki.}$$

$$\vec{p}'(1) = k_n \vec{a}_n \Rightarrow f'_{i,n}(1) = 0, \quad i = 0 \dots n-1, \\ f'_{n,n}(1) \neq 0.$$

5. oskulatorna ravnina u početnoj točki treba biti paralelna s  $\vec{a}_1$  i  $\vec{a}_2$

$$\Rightarrow f''_{1,n}(0) \neq 0, f''_{2,n}(0) \neq 0, \\ f''_{i,n}(0) = 0, \text{ ina\u0107e.}$$

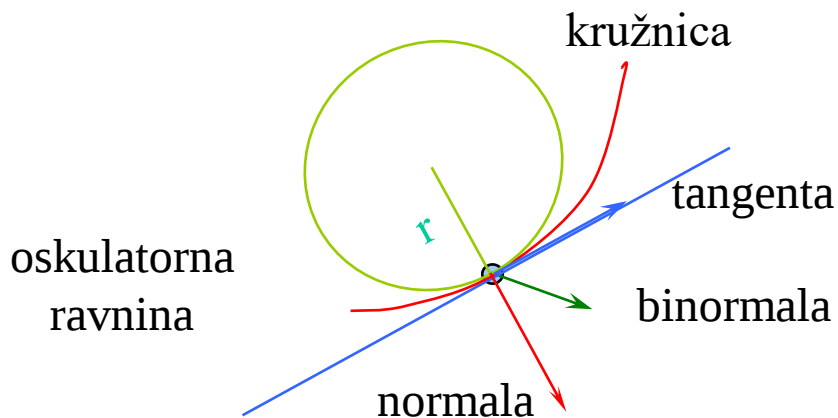
<http://people.math.aau.dk/~raussen/VIDIGEO/GEOLAB/apposcplane.html>

6. oskulatorna ravnina u završnoj točki treba biti paralelna s  $\vec{a}_{n-1}$  i  $\vec{a}_n$

$$\Rightarrow f''_{i,n}(1) = 0, \quad i = 0 \dots n-2, \\ f''_{n-1,n}(1) \neq 0, f''_{n,n}(1) \neq 0.$$

7. simetri\u010dnost te\u017einke funkcije - zamjena početne i završne to\u0107ke povla\u0107i promjenu smjera i redoslijeda vektora.

$$\text{oskulatorna} \Rightarrow f_{i,n}(t) = 1 - f_{n-i+1,n}(1-t), \quad i = 1 \dots n.$$



⇒ BEZIEROVE TEŽINSKE FUNKCIJE

$$f_{i,n}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_n(t)}{d^{(i-1)}t}, \quad \Phi_n(t) = \frac{1-(1-t)^n}{-t},$$

gdje  $d^{(i-1)}$  je  $(i-1)$  derivacija  $i = 1..n$

za derivaciju  $(-1)$  se podrazumijeva vrijednost 1

rekurzivni oblik pogodan za implementaciju na računalu:

$$f_{i,n}(t) = (1-t)f_{i,n-1}(t) + t f_{i-1,n-1}(t),$$

uvjeti zaustavljaju rekurzije

$$f_{0,0}(t) = 1, \quad f_{k+1,k}(t) = 0, \quad f_{-1,k}(t) = 1.$$

## \* PRIMJER

Odrediti Bezierove težinske funkcije ako su zadane četiri točke.

$$\Phi_3(t) = \frac{1 - (1-t)^3}{-t} = -3 + 3t - t^2,$$

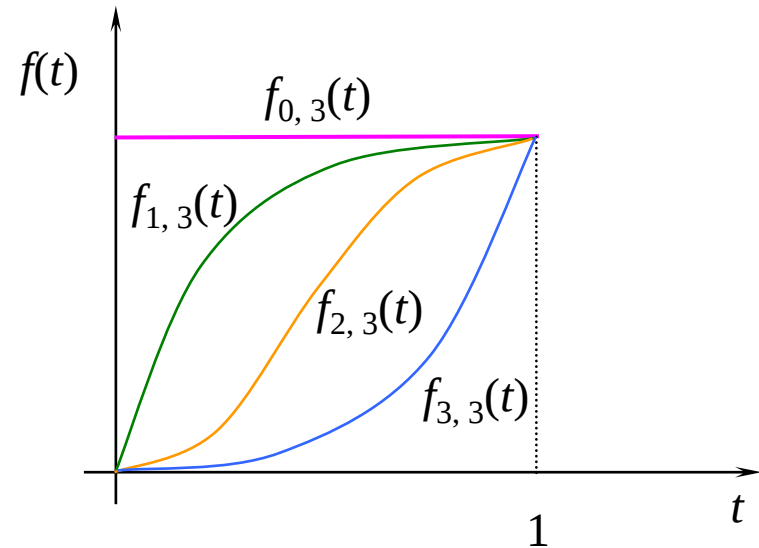
$$f_{i,3}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_3(t)}{d^{(i-1)}t},$$

$$f_{0,3}(t) = 1,$$

$$f_{1,3}(t) = 3t - 3t^2 + t^3,$$

$$f_{2,3}(t) = 3t^2 - 2t^3,$$

$$f_{3,3}(t) = t^3.$$



$$\vec{p}(t) = \sum_{i=0}^3 \vec{a}_i f_{i,3}(t)$$

$$\vec{p}(t) = \vec{a}_0 + (3t - 3t^2 + t^3)\vec{a}_1 + (3t^2 - 2t^3)\vec{a}_2 + t^3\vec{a}_3$$

Provjera postavljenih uvjeta na težinsku funkciju:

1. početna točka  $\vec{p}(0) = \vec{a}_0$
2. završna točka  $\vec{p}(1) = \vec{a}_0 + \vec{a}_1 + \vec{a}_2 + \vec{a}_3$

$$\vec{p}'(t) = (3 - 6t + 3t^2)\vec{a}_1 + (6t - 6t^2)\vec{a}_2 + 3t^2\vec{a}_3$$

3. derivacija u početnoj točki  $\vec{p}'(0) = 3\vec{a}_1$

4. derivacija u završnoj točki  $\vec{p}'(1) = 3\vec{a}_3$

$$\vec{p}''(t) = (-6 + 6t)\vec{a}_1 + (6 - 12t)\vec{a}_2 + 6t\vec{a}_3$$

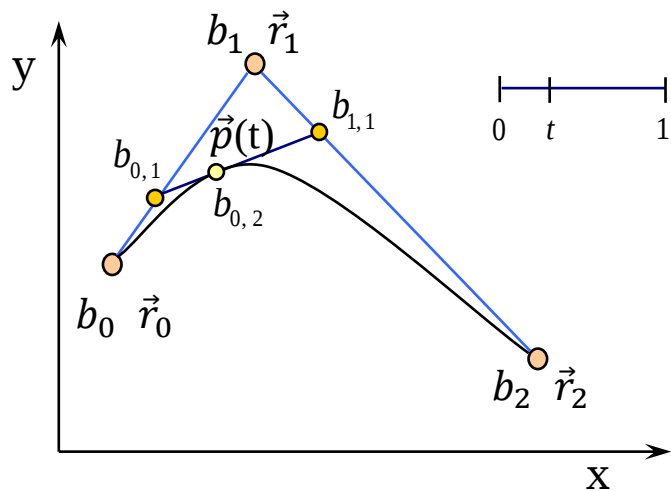
5. druga derivacija u početnoj točki  $\vec{p}''(0) = 6(\vec{a}_2 - \vec{a}_1)$

6. druga derivacija u završnoj točki  $\vec{p}''(1) = 6(\vec{a}_3 - \vec{a}_2)$

7. simetričnost  $f_{1,3}(t) = 1 - f_{3,3}(1 - t)$

## b) BERNSTEINOVE TEŽINSKE FUNKCIJE ZA KRIVULJU BEZIERA

### De Casteljaou - intuitivna geometrijska konstrukcija

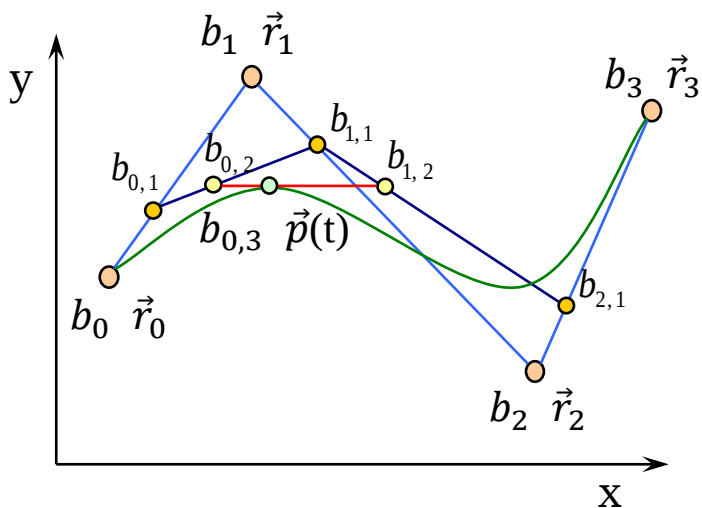


- uzastopne linearne interpolacije :

$$\left. \begin{aligned} b_{0,1} &= (1-t)b_0 + tb_1, \\ b_{1,1} &= (1-t)b_1 + tb_2, \end{aligned} \right\} b_{0,2} = (1-t)b_{0,1} + tb_{1,1},$$

- uvrstimo :

$$b_{0,2} = (1-t)^2 b_0 + 2t(1-t)b_1 + t^2 b_2$$



<http://www.martin-kraus.org/LiveGraphics3D/examples/parametrized/cagd/chap4fig1.html>

<https://www.jasondavies.com/animated-bezier/>

<http://www.saltire.com/HTML5/Advanced/Cubic%20Spline%20Curve.html>

<https://cagd-applets.webarchiv.kit.edu/mocca/html/noplugin/BezierCurve/AppDeCasteljau/index.html>

<http://www.kevs3d.co.uk/dev/scratchpad/>

(Bezier Cloud)

- poopćenje ovog postupka daje algoritam De Casteljaui

$$b_{i,j} = (1-t) b_{i,j-1} + t b_{i+1,j-1}(t), \quad i = 0..n-j, j = 1..n,$$

$$b_{i,0}(t) = b_i, \quad \vec{r}_i = b_i \text{ vrhovi kontrolnog poligona,}$$

$$b_{0,n}(t) = \vec{p}(t) \quad \text{točka na krivulji}$$

$$\vec{p}(t) = \sum_{i=0}^n \vec{r}_i b_{i,n}(t), \quad t \in [0, 1]$$

$$\text{vrijedi} \quad \sum_{i=0}^n b_{i,n}(t) = 1 \quad t \in [0, 1]$$

$b_{i,n}(t)$  – bazne funkcije – Bernsteino vi polinomi stupnja  $n$

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

Diskretna binomna razdioba:

$t$  – vjerojatnost događaja u svakom od  $n + 1$  pokušaja

$b_{i,n}$  – vjerojatnost postizanja točno  $i$  događaja u  $n + 1$  pokušaja

Binomni teorem: 
$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

### \* PRIMJER

Odrediti Bernsteinove težinske funkcije ako su zadane četiri točke.

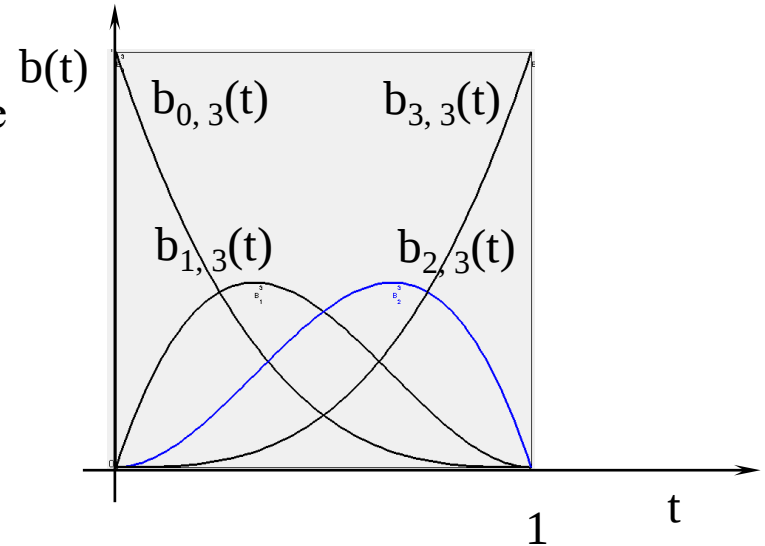
$$b_{i,3}(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i},$$

$$b_{0,3}(t) = (1-t)^3,$$

$$b_{1,3}(t) = 3t(1-t)^2,$$

$$b_{2,3}(t) = 3t^2(1-t),$$

$$b_{3,3}(t) = t^3.$$



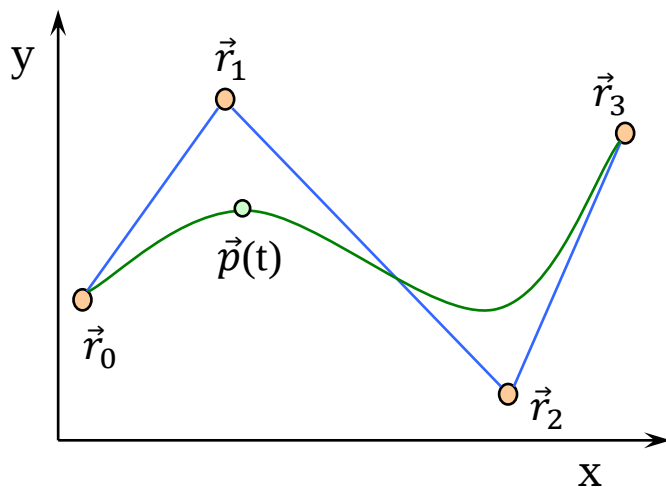
<http://graphics.cs.brown.edu>

<https://www.ibiblio.org/e-notes/Splines/bezier.html>

$$\vec{p}(t) = \sum_{i=0}^3 \vec{r}_i b_{i,3}(t),$$

$$\vec{p}(t) = (1-t)^3 \vec{r}_0 + 3t(1-t)^2 \vec{r}_1 + 3t^2(1-t) \vec{r}_2 + t^3 \vec{r}_3,$$

$$\begin{aligned} \vec{p}'(0) &= 3(\vec{r}_1 - \vec{r}_0), \\ \vec{p}'(1) &= 3(\vec{r}_3 - \vec{r}_2) \end{aligned}$$



## Krivulja Beziera – aproksimacijski oblik

a) Bezierov oblik Bezierove krivulje

$$\vec{p}(t) = \sum_{i=0}^n \vec{a}_i f_{i,n}(t), \quad t \in [0, 1]$$



$a_i$  kontrolni vektori

$$f_{i,n}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_n(t)}{d^{(i-1)}t},$$

$$\Phi_n(t) = \frac{1 - (1-t)^n}{-t}$$

b) Bernsteinov oblik Bezierove krivulje  
(de Casteljaou)

$$\vec{p}(t) = \sum_{i=0}^n \vec{r}_i b_{i,n}(t), \quad t \in [0, 1]$$



$r_i$  kontrolne točke

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

<https://webglfundamentals.org/webgl/lessons/resources/bezier-curve-diagram.html>

<http://bezier.method.ac/>

Matrično Bernsteinov oblik kubne Bezierove krivulje :

$$\vec{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_0 \\ \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}$$

Općenito, za tangentu na Bezierovu krivulju opisanu preko Bernsteinovih težinskih funkcija vrijedi:

$$\vec{p}'(0) = n (\vec{r}_1 - \vec{r}_0)$$

$$\vec{p}'(1) = n (\vec{r}_n - \vec{r}_{n-1}), \quad n \text{ .. stupanj krivulje}$$

Veza Bezierovih i Bernsteinovih težinskih funkcija (otkrio R. Forest):

$$f_{i,n}(t) = \sum_{j=i}^n b_{j,n}(t), \quad i = 0..n.$$

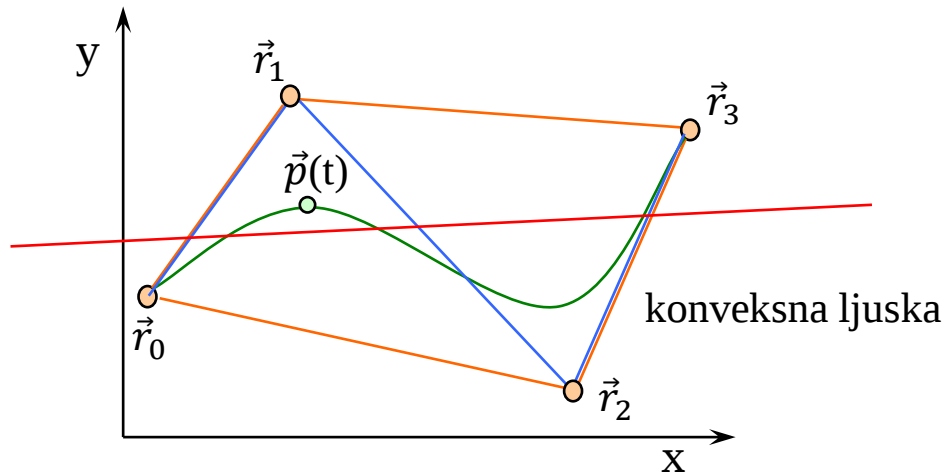
$$\vec{a}_0 = \vec{r}_0, \vec{a}_i = \vec{r}_i - \vec{r}_{i-1}, \quad i = 1..n,$$

ili

$$\vec{r}_0 = \vec{a}_0, \vec{r}_i = \vec{a}_i + \vec{r}_{i-1}, \quad i = 1..n,$$

# SVOJSTVA APROKSIMACIJSKIH BEZIEROVIH KRIVULJA

- postoji konveksna ljuska

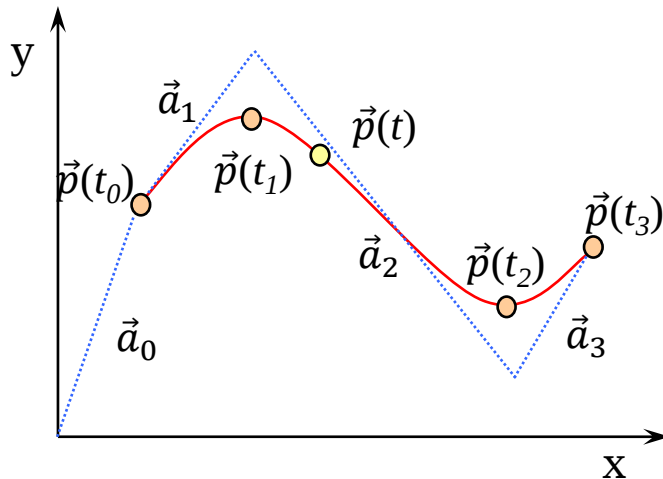


suma težinskih funkcija je 1  
važno kod ispitivanja sjecišta krivulje  
(težinske funkcije su nenegativne  
imaju točno jedan maksimum)

- krivulja nema više valova od kontrolnog poligona
- broj sjecišta ravnine i kontrolnog poligona  $\geq$  br. sjecišta ravnine i krivulje
- lokalni nadzor - nije ispunjeno
- broj kontrolnih točaka je u direktnoj vezi sa stupnjem krivulje
  - za  $n$  točaka stupanj je  $(n-1)$
- neovisnost o transformacijama (translacija, rotacija, skaliranje)
- simetričnost - kod uvrštenja možemo simetrično zamijeniti popis točaka
- <https://pomax.github.io/bezierjs/#getLUT>
- <https://webglfundamentals.org/webgl/lessons/resources/bezier-curve-diagram.html>
- [http://www.ibiblio.org/e-notes/Splines/water/little\\_game.html](http://www.ibiblio.org/e-notes/Splines/water/little_game.html)

## 6.3.2. INTERPOLACIJSKE KRIVULJE BEZIERA

Prolaze svim zadanim točkama. [Bezier](#)



$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^n \vec{a}_i f_{i,n}(t), \quad t \in [0, 1]$$

$f_{i,n}$  - poznato na osnovi broja točaka  
 $\vec{a}_i$  - nepoznato – određuje se na temelju  
 zadanih uvjeta (točaka kroz koje krivulja  
 prolazi, derivacija, zakrivljenosti i sl.)

Potreban je  $n+1$  uvjet.

Npr. Poznato je:

1.  $n+1$  točka krivulje s vrijednošću parametra  $\vec{p}_i(t_i), t_i = \frac{i}{n}, i = 0 \dots n$

ili

2. tangente u pojedinim točkama

$$\vec{p}'_i(t_i) = \sum_{i=1}^n \vec{a}_i f'_{i,n}(t),$$

ili

3. oskulatorne ravnine, položaji centara zakrivljenosti  $\vec{p}''_i(t_i) = \sum_{i=1}^n \vec{a}_i f''_{i,n}(t),$

## INTERPOLACIJSKA KRIVULJA KROZ $n+1$ TOČKU:

Neka su poznate točke:  $\overrightarrow{p_0} = \vec{p}(t_0), \overrightarrow{p_1} = \vec{p}(t_1), \overrightarrow{p_2} = \vec{p}(t_2), \dots, \overrightarrow{p_n} = \vec{p}(t_n)$   
uz parametar  $t_i = \frac{i}{n}$ , gdje je  $i = 0 \dots n$ .

$$\vec{p}(t) = \overrightarrow{a_0} + \sum_{i=1}^n \overrightarrow{a_i} f_{i,n}(t), \quad t \in [0, 1]$$

$$\begin{bmatrix} \overrightarrow{p_0} \\ \overrightarrow{p_1} \\ \overrightarrow{p_2} \\ \dots \\ \overrightarrow{p_n} \end{bmatrix} = \begin{bmatrix} 1 & f_{1,n}(0) & f_{2,n}(0) & \dots & f_{n,n}(0) \\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \dots & f_{n,n}(t_1) \\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \dots & f_{n,n}(t_2) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & f_{1,n}(1) & f_{2,n}(1) & \dots & f_{n,n}(1) \end{bmatrix} \begin{bmatrix} \overrightarrow{a_0} \\ \overrightarrow{a_1} \\ \overrightarrow{a_2} \\ \dots \\ \overrightarrow{a_n} \end{bmatrix}$$

uvrstili smo :  $t_0 = 0, t_n = 1$ .

uvrstit ćemo:

za početnu točku  $\vec{p}(0) = \vec{a}_0$ , tj.  $f_{0,n}(0)=1$ ,  $f_{i,n}(0) = 0, i = 1..n$ ,

za završnu točku  $\vec{p}(1) = \sum_{i=0}^n \vec{a}_i$ , tj.  $f_{i,n}(1)=1$ ,  $i = 0..n$ ,

$$\begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \dots \\ \vec{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \dots & f_{n,n}(t_1) \\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \dots & f_{n,n}(t_2) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \dots \\ \vec{p}_n \end{bmatrix}$$

kada odredimo nepoznate vektore  $\vec{a}_i$  možemo do pojedine točke krivulje doći na osnovi Bezierovih (korištenjem vektora  $\vec{a}_i$ ) ili Bernsteinovih (korištenjem vektora  $\vec{r}_i$ ) težinskih funkcija.

\* PRIMJER

Odrediti Interpolacijsku Bezierovu krivulju kroz četiri točke korištenjem Bezierovih težinskih funkcija.

Neka su poznate točke:  $\vec{p}_0 = \vec{p}(0), \vec{p}_1 = \vec{p}(1/3), \vec{p}_2 = \vec{p}(2/3), \vec{p}_3 = \vec{p}(1)$ ,

$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^3 \vec{a}_i f_{i,3}(t), \quad t \in [0, 1]$$

Iz prethodnog primjera za aproksimacijsku kubnu Bezierovu krivulju poznate su težinske funkcije:

$$\begin{aligned} f_{0,3}(t) &= 1, \\ f_{1,3}(t) &= 3t - 3t^2 + t^3, \\ f_{2,3}(t) &= 3t^2 - 2t^3, \\ f_{3,3}(t) &= t^3. \end{aligned} \quad \begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 19/27 & 7/27 & 1/27 \\ 1 & 26/27 & 20/27 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}$$

$$\vec{p}(t) = \vec{a}_0 + (3t - 3t^2 + t^3)\vec{a}_1 + (3t^2 - 2t^3)\vec{a}_2 + t^3\vec{a}_3$$

<http://www.redblobgames.com/articles/curved-paths/>

### 6.3.3. RAZLOMLJENE FUNKCIJE

#### PRIKAZ KRIVULJA POMOĆU KVADRATNIH RAZLOM. FUNKCIJA

- pogodan oblik za prikaz krivulja drugog reda
- homogena koordinata omogućava prikaz koničnih krivulja (presjek ravnine i stošca) <https://observablehq.com/@mcmclur/conic-sections>
- invarijantnost na transformaciju perspektivne projekcije (nerazlomljene krivulje su invarijantne samo na translaciju, rotaciju, skaliranje)

$$\left. \begin{aligned} x &= \frac{x_1}{x_4} = \frac{a_1 t^2 + b_1 t + c_1}{at^2 + bt + c}, \\ y &= \frac{x_2}{x_4} = \frac{a_2 t^2 + b_2 t + c_2}{at^2 + bt + c}, \\ z &= \frac{x_3}{x_4} = \frac{a_3 t^2 + b_3 t + c_3}{at^2 + bt + c} \end{aligned} \right\} \begin{array}{l} \text{u radnom} \\ \text{prostoru} \end{array}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \end{bmatrix} \quad \text{matrični oblik}$$

$\mathbf{K}$  - karakteristična matrica kvadratne krivulje,  $0 \leq t \leq 1$ ,

$$\mathbf{X} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \mathbf{K}$$

- derivacije vektora  $[x_1 \ x_2 \ x_3 \ x_4]$  po parametru  $t$  - u homogenom prostoru
- matrica  $\mathbf{K}$  određuje i derivacije duž krivulje

$$x'_1 = \frac{d x_1}{d t} = 2a_1 t + b_1,$$

$$x'_2 = \frac{d x_2}{d t} = 2a_2 t + b_2,$$

$$x'_3 = \frac{d x_3}{d t} = 2a_3 t + b_3,$$

$$x'_4 = \frac{d x_4}{d t} = 2a t + b.$$

$$\mathbf{X}' = \begin{bmatrix} x'_1 & x'_2 & x'_3 & x'_4 \end{bmatrix} = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}$$

$$\mathbf{X}'' = \begin{bmatrix} x''_1 & x''_2 & x''_3 & x''_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}$$

$$\mathbf{X}' = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}$$

[https://mathinsight.org/parametrized\\_curve\\_derivative](https://mathinsight.org/parametrized_curve_derivative)

$$\mathbf{X}'' = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}$$

♣ kvadratna razlomljena krivulja određena je s tri točke

$$\mathbf{V}_0, \quad t_0 = 0, \quad \mathbf{X}_0 = [\mathbf{V}_0 \quad 1]$$

$$\mathbf{V}_1, \quad t_1 = \frac{1}{2}, \quad \mathbf{X}_1 = [\mathbf{V}_1 \quad 1]$$

$$\mathbf{V}_2, \quad t_2 = 1, \quad \mathbf{X}_2 = [\mathbf{V}_2 \quad 1]$$

tri točke uvrstimo u jednadžbu krivulje,  
uzmimo da su poznati iznosi parametra

$$\mathbf{K} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix}^{-1} \mathbf{X}$$

$$\mathbf{K} = \begin{bmatrix} t_0^2 & t_0 & 1 \\ t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

\* PRIMJER

Neka su zadane tri točke

i pripadni iznosi parametra.

Odrediti kvadratnu razlomljenu krivulju.

$$\mathbf{V}_0, \quad t_0=0, \quad \mathbf{X}_0 = \begin{bmatrix} r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_1, \quad t_1=\frac{1}{2}, \quad \mathbf{X}_1 = \begin{bmatrix} 0 & r & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_2, \quad t_2=1, \quad \mathbf{X}_2 = \begin{bmatrix} -r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & 1 \\ -r & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

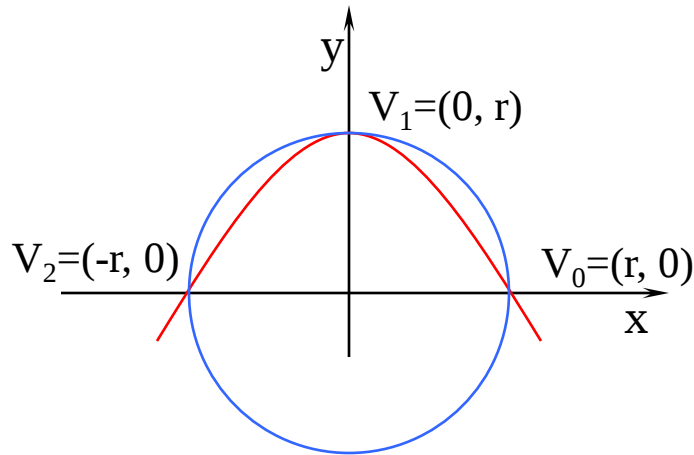
$$x_1 = -2rt + r = r(1-2t),$$

$$x_2 = -4rt^2 + 4rt = 4r(t-t^2),$$

$$x_3 = 0,$$

$$x_4 = 1.$$

po komponentama



Rezultat je parabola, to je opća krivulja drugog reda.

Ako želimo načiniti kružnicu potrebno je upotrijebiti analitičke poznate izraze za kružnicu.

$$\left. \begin{array}{l} x = r \cos \varphi \\ y = r \sin \varphi \\ t = \operatorname{tg} \frac{\varphi}{2} \end{array} \right\} \left. \begin{array}{l} x = r \frac{1-t^2}{1+t^2} \\ y = r \frac{2t}{1+t^2} \end{array} \right\} \begin{array}{l} x_1 = r(1-t^2), \\ x_2 = 2rt, \\ x_3 = 0, \\ x_4 = 1+t^2. \end{array}$$

$$\mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad x_4] = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

## PRIKAZ KRIVULJA POMOĆU KUBNIH RAZLOMLJENIH FUNKCIJA

- kvadratnim razlomljenim funkcijama ne možemo prikazati infleksiju i ostale pojave višeg reda

$$\left. \begin{aligned} x &= \frac{x_1}{x_4} = \frac{a_1 t^3 + b_1 t^2 + c_1 t + d_1}{a t^3 + b t^2 + c t + d}, \\ y &= \frac{x_2}{x_4} = \frac{a_2 t^3 + b_2 t^2 + c_2 t + d_2}{a t^3 + b t^2 + c t + d}, \\ z &= \frac{x_3}{x_4} = \frac{a_3 t^3 + b_3 t^2 + c_3 t + d_3}{a t^3 + b t^2 + c t + d} \end{aligned} \right\} \begin{array}{l} \text{u radnom} \\ \text{prostoru} \end{array}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \\ d_1 & d_2 & d_3 & d \end{bmatrix} \quad \text{matrični oblik}$$

$\mathbf{A}$  - karakteristična matrica kubne krivulje,  $0 \leq t \leq 1$ ,

$$\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}$$

- derivacije vektora  $[x_1 \ x_2 \ x_3 \ x_4]$  po parametru  $t$  - u homogenom prostoru
- matrica  $\mathbf{A}$  određuje i derivacije duž krivulje

$$x'_1 = \frac{d x_1}{d t} = 3a_1 t^2 + 2b_1 t + c_1,$$

$$x'_2 = \frac{d x_2}{d t} = 3a_2 t^2 + 2b_2 t + c_2,$$

$$x'_3 = \frac{d x_3}{d t} = 3a_3 t^2 + 2b_3 t + c_3,$$

$$x'_4 = \frac{d x_4}{d t} = 3a t^2 + 2b t + c.$$

$$\mathbf{X}' = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$\mathbf{X}'' = \begin{bmatrix} 6t & 2 & 0 & 0 \end{bmatrix} \mathbf{A}$$

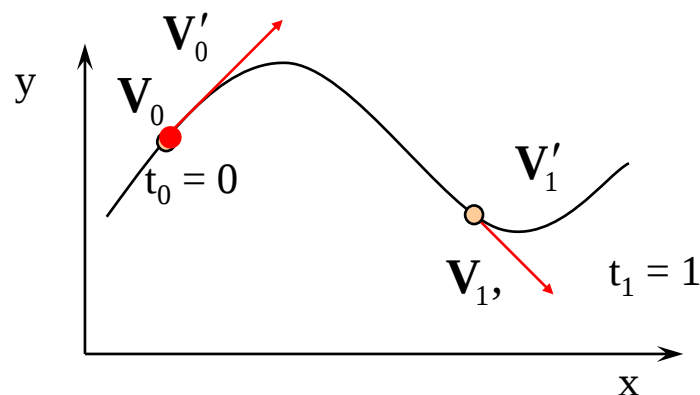
$$\mathbf{X}''' = \begin{bmatrix} 6 & 0 & 0 & 0 \end{bmatrix} \mathbf{A}$$

- za određivanje kubne razlomljene krivulje potrebna su četiri uvjeta (kako bi mogli invertirati matricu)

To mogu biti 4 točke ili na primjer 2 točke i 2 derivacije.

<http://www.theparticle.com/applets/nyu/BezierApplet/>

- ♣ kubna razlomljena krivulja određena s dvije rubne točke i derivacije



$$\mathbf{X} = (x_1 \quad x_2 \quad x_3 \quad x_4) = [x_4 x \quad x_4 y \quad x_4 z \quad x_4] = [t^3 \quad t^2 \quad t \quad 1] \mathbf{A}$$

$$\mathbf{X}' = (x'_1 \quad x'_2 \quad x'_3 \quad x'_4) = \left[ (x_4 x)' \quad (x_4 y)' \quad (x_4 z)' \quad x'_4 \right] = [3t^2 \quad 2t \quad 1 \quad 0] \mathbf{A}$$

$$t_0 = 0, \quad \mathbf{X}_0 = [x_{40} \mathbf{V}_0 \quad x_{40}] = \begin{bmatrix} t_0^3 & t_0^2 & t_0 & 1 \end{bmatrix} \mathbf{A} = [0 \quad 0 \quad 0 \quad 1] \mathbf{A}$$

$$t_1 = 1, \quad \mathbf{X}_1 = [x_{41} \mathbf{V}_1 \quad x_{41}] = \begin{bmatrix} t_1^3 & t_1^2 & t_1 & 1 \end{bmatrix} \mathbf{A} = [1 \quad 1 \quad 1 \quad 1] \mathbf{A}$$

$$t_0 = 0, \quad \mathbf{X}'_0 = [x'_{40} \mathbf{V}_0 + x_{40} \mathbf{V}'_0 \quad x'_{40}] = \begin{bmatrix} 3t_0^2 & 2t_0 & 1 & 0 \end{bmatrix} \mathbf{A} = [0 \quad 0 \quad 1 \quad 0] \mathbf{A}$$

$$t_1 = 1, \quad \mathbf{X}'_1 = [x'_{41} \mathbf{V}_1 + x_{41} \mathbf{V}'_1 \quad x'_{41}] = \begin{bmatrix} 3t_1^2 & 2t_1 & 1 & 0 \end{bmatrix} \mathbf{A} = [3 \quad 2 \quad 1 \quad 0] \mathbf{A}$$

$$\begin{bmatrix} x_{40} \mathbf{V}_0 & x_{40} \\ x_{41} \mathbf{V}_1 & x_{41} \\ x'_{40} \mathbf{V}_0 + x_{40} \mathbf{V}'_0 & x'_{40} \\ x'_{41} \mathbf{V}_1 + x_{41} \mathbf{V}'_1 & x'_{41} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_{40} \mathbf{V}_0 & x_{40} \\ x_{41} \mathbf{V}_1 & x_{41} \\ x'_{40} \mathbf{V}_0 + x_{40} \mathbf{V}'_0 & x'_{40} \\ x'_{41} \mathbf{V}_1 + x_{41} \mathbf{V}'_1 & x'_{41} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{M} \mathbf{H} \mathbf{V}$$

**M**.....univerzaln a transformacijska matrica

- Segment krivulje određen rubnim točkama i derivacijama u njima

$$\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \mathbf{M} \mathbf{H} \mathbf{V}$$

**M** - ne ovisi o obliku krivulje već o izboru točaka (derivacija)

**H** - krivulja prolazi početnom i krajnjom točkom uz zadane derivacije, a derivacija homogene komponente određuje kako će prolaziti

- ako je  $x'_{40} = x'_{41} = 0$  dobit ćemo specijalan slučaj odnosno običnu parametarsku kubnu krivulju koja se zove HERMITOVA KRIVULJA

**V** - zadane točke i derivacije koje određuju segment krivulje u radnom prostoru

$$\mathbf{M} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

- HERMITOVA KRIVULJA

$$\mathbf{X} = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

- VEZA HERMITOVE I BEZIEROVE KRIVULJE (preko Bernsteinovih polinoma)

$$\vec{p}(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_0 \\ \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} \text{ uz } \begin{aligned} \mathbf{V}_0 &= \vec{r}_0 \\ \mathbf{V}_1 &= \vec{r}_3 \\ \mathbf{V}'_0 &= 3(\vec{r}_1 - \vec{r}_0) \\ \mathbf{V}'_1 &= 3(\vec{r}_3 - \vec{r}_2) \end{aligned}$$

$\Rightarrow$  radi se o istoj krivulji

**\* PRIMJER**

Neka su zadane dvije točke  
i derivacije u njima.

Odrediti kubnu razlomljenu krivulju.

$$\mathbf{V}_0 = [0 \quad 0 \quad 0], \quad t_0 = 0,$$

$$\mathbf{V}_1 = [1 \quad 0 \quad 0], \quad t_1 = 1,$$

$$\mathbf{V}'_0 = [1 \quad 1 \quad 0], \quad t_0 = 0,$$

$$\mathbf{V}'_1 = [1 \quad -1 \quad 0], \quad t_1 = 1.$$

$$[x_{40} \quad x_{41} \quad x'_{40} \quad x'_{41}] = [1 \quad 1 \quad a \quad b]$$

$$\mathbf{A} = \mathbf{M}\mathbf{H}\mathbf{V} =$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 & 0 & a+b \\ -b & -1 & 0 & -(2a+b) \\ 1 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad x_4] = [t^3 \quad t^2 \quad t \quad 1] \mathbf{A}$$

$$\left. \begin{aligned} x &= \frac{x_1}{x_4} = \frac{bt^3 - bt^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ y &= \frac{x_2}{x_4} = \frac{-t^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ z &= \frac{x_3}{x_4} = 0 \end{aligned} \right\} a, b = ?$$

Uvodimo dodatnu točku  $V_2 = (1/2 \ 1/2 \ 0)$ ,  $t_2 = 1/2$ .  $\Rightarrow a = -2, \ b = 2$

<http://www.rose-hulman.edu/~finn/courses/MA323GeomModel/TestApplets/RationalC2Spline.html> +/- A/Z

$$x = \frac{2t^3 - 2t^2 + t}{2t^2 - 2t + 1},$$

$$y = \frac{-t^2 + t}{2t^2 - 2t + 1},$$

$$z = 0$$

## VEZA KOORDINATA I PARAMETARSKIH DERIVACIJA IZMEĐU RADNOG I HOMOGENOG PROSTORA

- radni prostor:

$$\mathbf{V}(t) = [x(t) \quad y(t) \quad z(t)], \quad \frac{d\mathbf{V}(t)}{dt} = \begin{bmatrix} \frac{x(t)}{dt} & \frac{y(t)}{dt} & \frac{z(t)}{dt} \end{bmatrix}.$$

- homogeni prostor

$$\mathbf{X}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)], \quad \frac{d\mathbf{X}(t)}{dt} = \begin{bmatrix} \frac{x_1(t)}{dt} & \frac{x_2(t)}{dt} & \frac{x_3(t)}{dt} & \frac{x_4(t)}{dt} \end{bmatrix}.$$

- VEZA KOORDINATA

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4}$$

$$x_1 = x x_4, \quad x_2 = y x_4, \quad x_3 = z x_4$$

$$\mathbf{X} = (x_1 \quad x_2 \quad x_3 \quad x_4) = [x_4 x \quad x_4 y \quad x_4 z \quad x_4] = x_4 [x \quad y \quad z \quad 1] = x_4 [\mathbf{V} \quad 1]$$

$$\mathbf{X} = x_4 [\mathbf{V} \quad 1]$$

- VEZA PRVE DERIVACIJE - homogena komponenta nije konstanta

$$\mathbf{X}' = (x'_1 \quad x'_2 \quad x'_3 \quad x'_4) = \begin{bmatrix} (x_4 x)' & (x_4 y)' & (x_4 z)' & x'_4 \end{bmatrix} =$$

$$\begin{bmatrix} (x'_4 x + x_4 x') & (x'_4 y + x_4 y') & (x'_4 z + x_4 z') & x'_4 \end{bmatrix} = \begin{bmatrix} x'_4 & x_4 \end{bmatrix} \begin{bmatrix} x & y & z & 1 \\ x' & y' & z' & 0 \end{bmatrix}$$

$$\mathbf{X}' = \begin{bmatrix} x'_4 & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix}$$

- VEZA DRUGE DERIVACIJE

$$\mathbf{X}'' = (\mathbf{X}')' = \left( \begin{bmatrix} x'_4 & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix} \right)' = ((x'_4 \mathbf{V} + x_4 \mathbf{V}') \quad x'_4)' = \begin{bmatrix} (x_4 \mathbf{V})'' & x_4'' \end{bmatrix} =$$

$$= \begin{bmatrix} (x_4'' \mathbf{V} + 2 x'_4 \mathbf{V}' + x_4 \mathbf{V}'') & x_4'' \end{bmatrix}$$

$$\mathbf{X}'' = \begin{bmatrix} x_4'' & 2x'_4 & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \\ \mathbf{V}'' & 0 \end{bmatrix}$$

\* PRIMJER

Odrediti prvu derivaciju u homogenom i radnom prostoru na kružnicu.

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

homogeni prostor :

$$x_1 = r(1 - t^2),$$

$$x_2 = 2rt,$$

$$x_4 = t^2 + 1.$$

$$x'_1 = -2rt,$$

$$x'_2 = 2r,$$

$$x'_4 = 2t.$$

radni prostor :

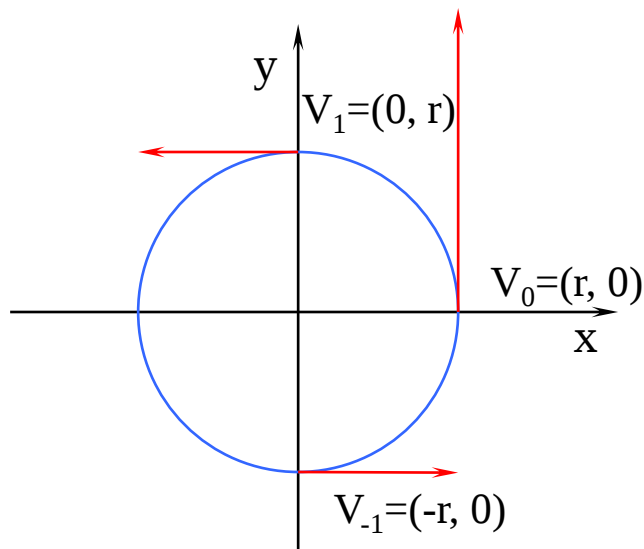
$$x = \frac{r(1 - t^2)}{1 + t^2},$$

$$y = \frac{2rt}{1 + t^2}.$$

$$x' = \left( \frac{x_1}{x_4} \right)' = \frac{-4rt}{1 + 2t^2 + t^4} \neq \frac{x'_1}{x'_4},$$

$$y' = \left( \frac{x_2}{x_4} \right)' = \frac{2r(1 - t^2)}{1 + 2t^2 + t^4} \neq \frac{x'_2}{x'_4}.$$

| t  | x <sub>1</sub> | x <sub>2</sub> | x <sub>4</sub> | x | y  | x' <sub>1</sub> | x' <sub>2</sub> | x' <sub>4</sub> | x' | y' |
|----|----------------|----------------|----------------|---|----|-----------------|-----------------|-----------------|----|----|
| 0  | r              | 0              | 1              | r | 0  | 0               | 2r              | 0               | 0  | 2r |
| 1  | 0              | 2r             | 2              | 0 | r  | -2r             | 2r              | 2               | -r | 0  |
| -1 | 0              | -2r            | 2              | 0 | -r | 2r              | 2r              | -2              | r  | 0  |



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \quad \text{nagib tangente, gdje je } \dot{x} = dx/dt$$

$$\frac{d^2y}{d^2x} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} \frac{dt}{dx} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3} \quad \text{zakrivljenost}$$

<http://people.math.aau.dk/~raussen/VIDIGEO/GEOLAB/speed.html>

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|x' \cdot y'' - y' \cdot x''|}{\sqrt{(x')^2 + (y')^2}^3} = \frac{1}{r} \quad \text{zakrivljenost (curvature)}$$

## 6.3.4 POVRŠINE

- promatramo geometrijsko mjesto točaka (trag) koji nastaje ako se neka krivulja pomiče i istovremeno deformira u prostoru
- segment površine čini krpicu

povezivanje krpica uz ostvarivanje kontinuiteta duž spojeva

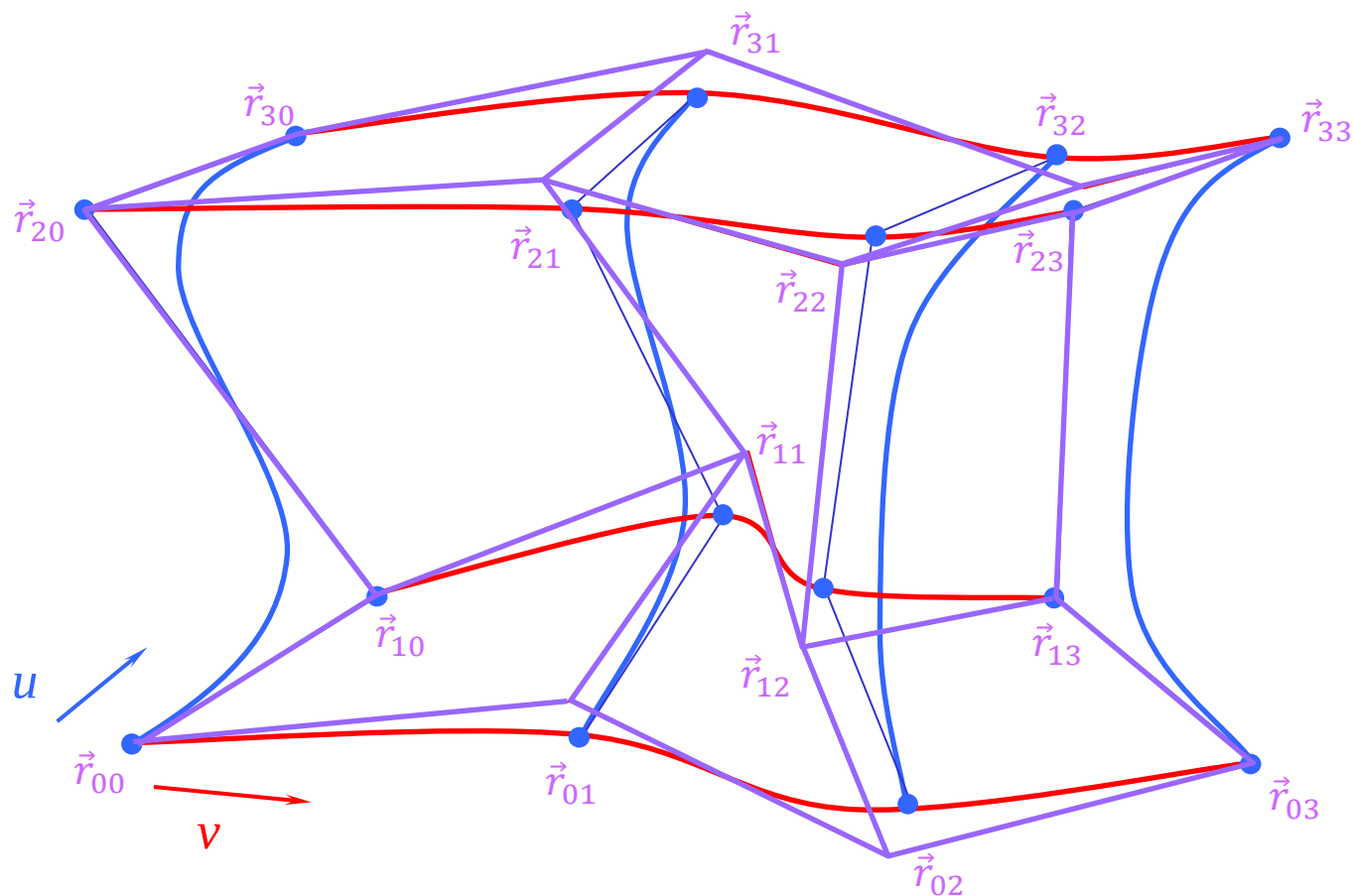
- $C^0$  jednakost točaka krivulja duž spoja
- $C^1$  iste parcijalne derivacije (poprečno) <http://stemkoski.github.io/MathBox/graph3d-deriv.html>
- $C^2$  zakrivljenost (poprečno) [http://profs.sci.univr.it/~baldo/tjs/principal\\_curvatures.html](http://profs.sci.univr.it/~baldo/tjs/principal_curvatures.html)

## POVRŠINA BEZIERA

- ako koristimo krivulje Beziera - dobit ćemo krpicu [Beziera](#), 2.
- GENERATRISA - generira površinu
- DIREKTRISA - krivulje koje određuju kako će generatrisa gibati kroz prostor
- tenzorski produkt dvije krivulje

- površina je funkcija dva parametra  $u, v$

<http://acko.net/files/mathbox/MathBox.js/examples/BezierSurface.html>

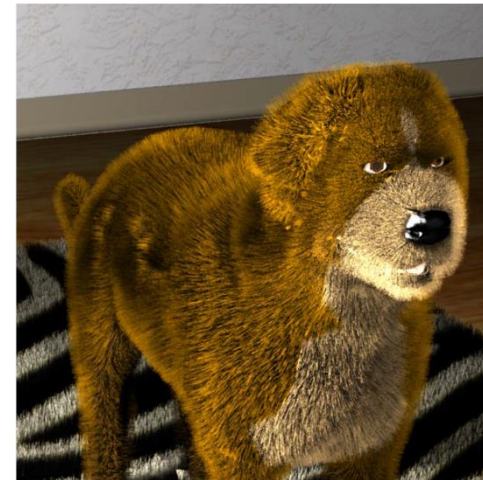


- generatriše <https://www.ibiblio.org/e-notes/Splines/bezier3d.html>
- direktrise

- prva i zadnja direktrisa zovu se glavne direktrise jer kontrolne točke generatriše leže na njima
- putanje vrhova kontrolnih poligona generatriše određuju direktrise  
 $\Rightarrow$  direktrise ne leže na površini (osim glavnih direktrisa)
- stupanj krivulja određuje krpicu

npr. bikvadratne, bikubične [https://www.ibiblio.org/e-notes/Splines/models/eagle\\_anim.html](https://www.ibiblio.org/e-notes/Splines/models/eagle_anim.html)  
<https://amarnaths0005.github.io/3DCurvesSurfaces/#four>

$$\vec{p}(u, v) = [u^3 \quad u^2 \quad u \quad 1] \mathbf{M} \begin{bmatrix} \mathbf{D}_0(\mathbf{v})^\tau \\ \mathbf{D}_1(\mathbf{v})^\tau \\ \mathbf{D}_2(\mathbf{v})^\tau \\ \mathbf{D}_3(\mathbf{v})^\tau \end{bmatrix}$$



vektori normala na površinu

- pojedine direktrise

$$D_0(v) = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} M \begin{bmatrix} r_{00} \\ r_{01} \\ r_{02} \\ r_{03} \end{bmatrix} \quad D_1(v) = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} M \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

$$D_2(v) = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} M \begin{bmatrix} r_{20} \\ r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} \quad D_3(v) = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} M \begin{bmatrix} r_{30} \\ r_{31} \\ r_{32} \\ r_{33} \end{bmatrix}$$

- krpica površine

$$\vec{p}(u, v) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} r_{00} & r_{01} & r_{02} & r_{03} \\ r_{10} & r_{11} & r_{12} & r_{13} \\ r_{20} & r_{21} & r_{22} & r_{23} \\ r_{30} & r_{31} & r_{32} & r_{33} \end{bmatrix} \mathbf{M}^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

- Npr: <http://vladamakaric.github.io/InteractiveBezierSurface/>
- <https://serhangursoy.github.io/BezierSurfaceWithWebGL/>