4. Krivulje i površine

KRIVULJE

– analitički izraz izvorne krivulje u pravilu je nepoznat

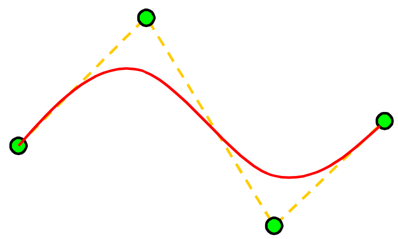
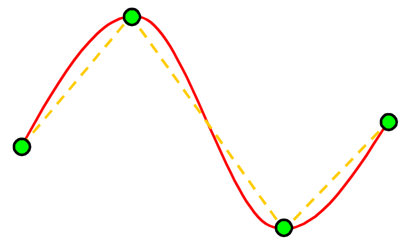
– poznato je
  - koordinate u nekim točkama
  - nagibi, zakrivljenost ili izvijanje u nekim točkama

  $\Rightarrow$ modeliranje

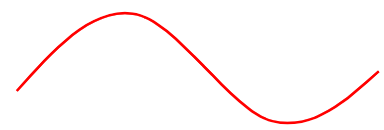
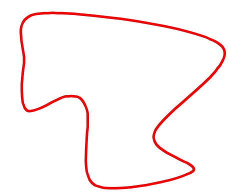
– opis segmenta krivulje
– segmentiranje
  - povezivanje segmenata uz ostvarivanje kontinuiteta između segmenata

PODJELA KRIVULJA

- aproksimacijske
- interpolacijske

- otvorene
- zatvorene

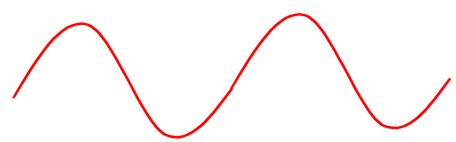
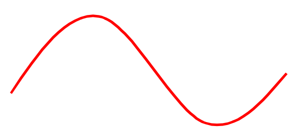
- razlomljene
- nerazlomljene

$$x(t)=\frac{a_1t^3+b_1t^2+c_1t+d_1}{at^3+bt^2+ct+d}$$

$$x(t)=a_1t^3+b_1t^2+c_1t+d_1$$

- periodične
- neperiodične

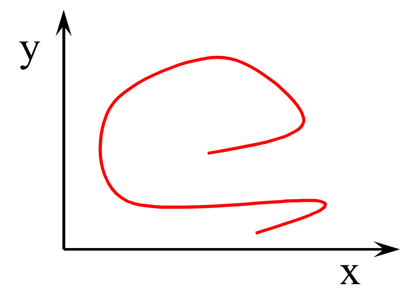
(periodičnost težinskih funkcija)

POŽELJNA SVOJSTVA KRIVULJA

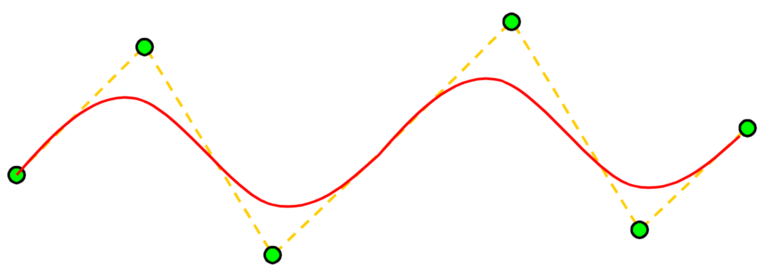
- višestruke vrijednosti

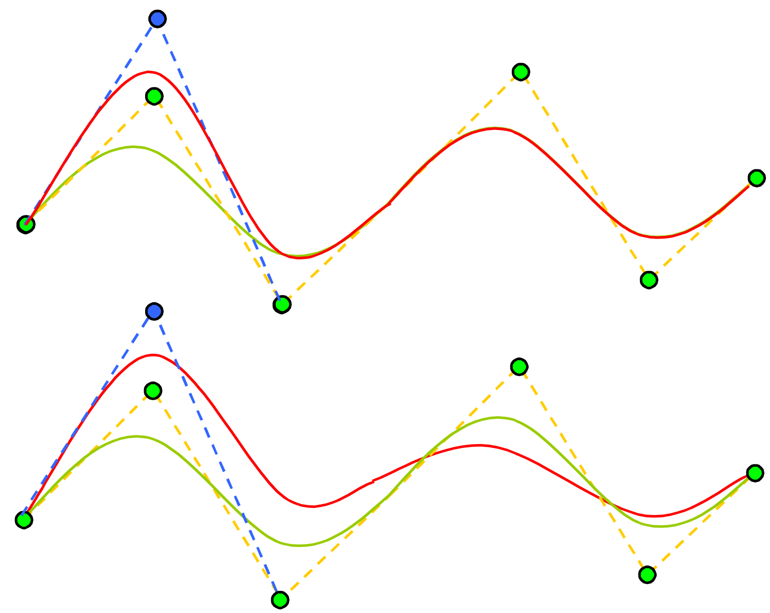
http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/3Dparametrization.html



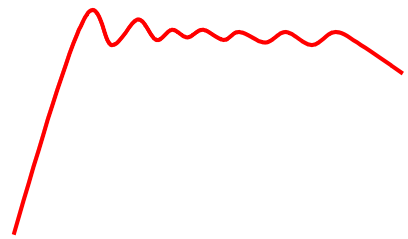
- neovisnost o koordinatnom sustavu (Kartezijev, polarni)

- lokalni nadzor

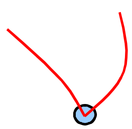




- smanjenje varijacije - kod visokog stupnja polinoma može se javiti titranje krivulje

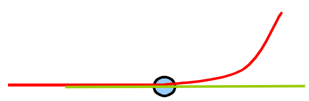


- kontrola reda neprekinutosti
- http://www.slu.edu/classes/maymk/Applets/Derivatives2.html



C^0

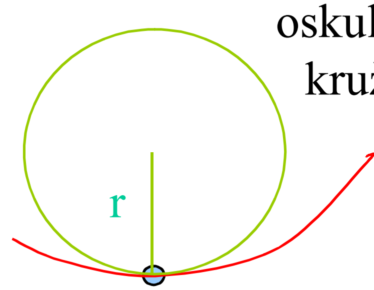
ista vrijednost koordinate



C^1

ista tangenta

oskulatorna kružnica

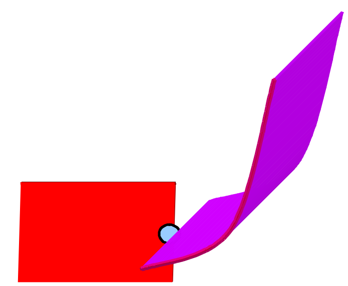


C^2

kontinuitet radijusa zakrivljenosti

http://www.ies.co.jp/math/java/calc/curve/curve.html

http://www.vis.uni-stuttgart.de/~kraus/LiveGraphics3D/cagd/chap10fig4.html



C^3

kontinuitet u izvijanju

C^0 - ista vrijednost koordinate $f(t) = g(t)$

C^1 - ista vrijednost derivacije $f'(t) = g'(t)$

C^2 - ista vrijednost druge derivacije $f''(t) = g''(t)$

Zakrivljenost krivulje obrnuto je proporcionalna radijusu oskulatorne kružnice. Ako je radijus velik zakrivljenost je mala (i obrnuto).

C^3 - ista vrijednost treće derivacije $f'''(t) = g'''(t)$

Osim C kontinuiteta postoje i G kontinuiteti koji zahtijevaju proporcionalnost.

G (geometrijski)

G^1 - proporcionalna vrijednost derivacije $f'(t) = k_1\, g'(t), k_1>0$

G^2 - proporcionalna vrijednost druge derivacije $f''(t) = k_2\, g''(t), k_2>0$

G^3 - proporcionalna vrijednost treće derivacije $f'''(t) = k_3\, g'''(t), k_3>0$

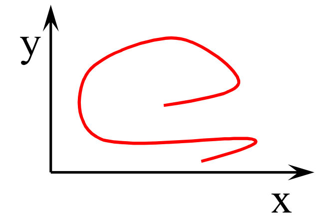
C^1 kontinuitet implicira G^1 kontinuitet osim kada je vektor tangente [0 0 0]

kod C^1 kontinuiteta može doći do promjene smjera, kod G^1 ne može.

ANALITIČKI OPIS PROSTORNIH KRIVULJA

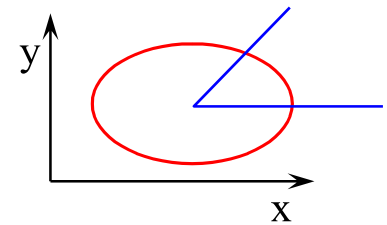
a) eksplicitni oblik - nemogućnost prikaza višestrukih vrijednosti

$$y = f(x), \qquad z = g(x)$$



b) implicitni oblik - za prikaz dijela krivulje trebaju dodatni uvjeti

$$F(x,\ y,\ z) = 0$$

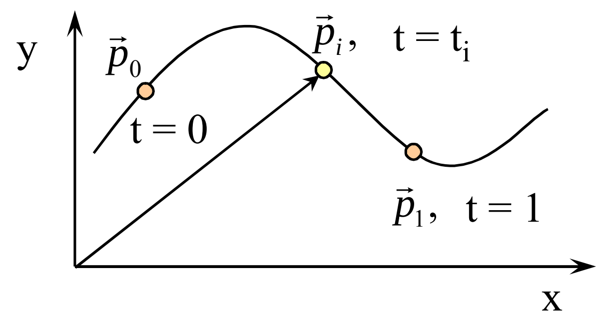


c) parametarski oblik

$$x = \mathrm{x}(t), \quad y = \mathrm{y}(t), \quad z = \mathrm{z}(t).$$

točka na krivulji - vektorska funkcija

$$\mathrm{V}(t) = [\mathrm{x}(t) \quad \mathrm{y}(t) \quad \mathrm{z}(t)].$$



vektor tangente

$$\mathrm{V}'(t) = [\mathrm{x}'(t) \quad \mathrm{y}'(t) \quad \mathrm{z}'(t)].$$

$$\mathrm{V}(t_i) = \vec{p}(t_i) = \vec{p}_{t_i}.$$

4.1 SEGMENT KRIVULJE

4.1.1. KRIVULJA BEZIERA

Postupak poznat pod imenom krivulje Beziera nezavisno su razvili

- BEZIER 1962. Rénault
- DE CASTELJAU 1959. Citroën

kao polaznu osnovu u CAD sustavima. De Casteljau direktno koristi Bernsteinove polinome.

1970. R. Forest otkriva vezu Bezierovog rada i Bernsteinovih polinoma.

P. Bezier objavljuje svoj rad i krivulje dobivaju ime po njemu.

- aproksimacijske krivulje Beziera
- interpolacijske krivulje Beziera

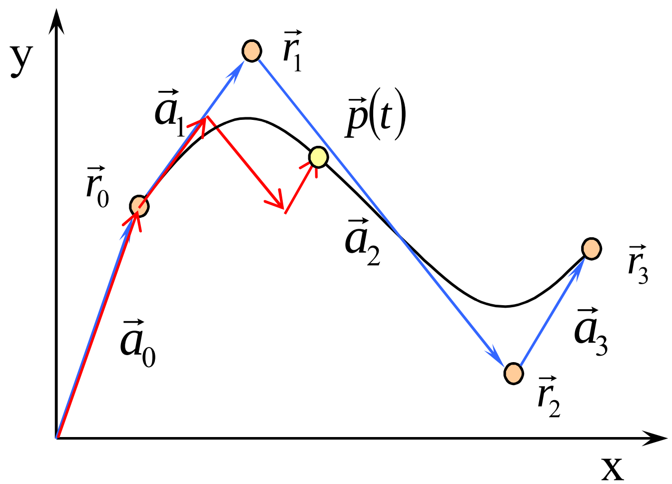
- Bezierove težinske funkcije (Bezier)
- Bernsteinove težinske funkcije (De Casteljau)

APROKSIMACIJSKE KRIVULJE BEZIERA

Prolaze početnom i krajnjom točkom, a ostalima se samo približava.

a) BEZIEROVE TEŽINSKE FUNKCIJE

Korištenje gibanja vrha sastavljenog otvorenog poligona.



$$\vec{p}(t) = \sum_{i=0}^{n} \vec{a}_i f_{i,n}(t) \qquad t \in [0, 1]$$

$$\vec{a}_0 = \vec{r}_0 ,$$

$$\vec{a}_i = \vec{r}_i - \vec{r}_{i-1}, \qquad i = 1..\, n$$

$n+1$ .. broj točaka.

n .. stupanj krivulje.

a_i .. kontrolni poligon.

$p(t)$ .. točka na krivulji - linearna kombinacija $f_{i,n}(t)$ i a_i.

$f_{i,n}(t)$ .. težinska funkcija - njena vrijednost pokazuje koliko i-ti element poligona pridonosi pripadnoj točki za parametar t.

$f_{i,n}(t)$ - težinska funkcija je općenita i mora zadovoljiti niz posebnih uvjeta:

1. <u>početna točka</u> $p(0) = a_0 \Rightarrow f_{0,n}(0) = 1,$
$f_{\mathrm{i},n}(0) = 0, \quad i = 1 \mathrel{..} n$

2. <u>završna točka</u> $p(1) = \Sigma a_i \Rightarrow f_{i,n}(1) = 1, \quad i = 0 \mathrel{..} n$ zbroj svih vektora

3. <u>osnovni vektor</u> a_1 treba biti paralelan s tangentom u početnoj točki
$p'(0) = k_1 a_1 \Rightarrow f'_{1,\mathrm{n}}(0) \neq 0,$
$f'_{\mathrm{i},n}(0) = 0, \quad i \neq 1$

4. <u>osnovni vektor</u> a_n treba biti paralelan s tangentom u završnoj točki.
$p'(1) = k_n a_n \Rightarrow f'_{i,n}(1) = 0, \quad i = 0 \mathrel{..} n\text{-}1,$
$f'_{n,n}(1) \neq 0.$

5. oskulatorna ravnina u početnoj točki treba biti paralelna s a_1 i a_2

$$\Rightarrow\ f''_{1,n}(0) \neq 0,\ \ f''_{2,n}(0) \neq 0,$$
$$f''_{i,n}(0) = 0,\ \ \text{inače.}$$

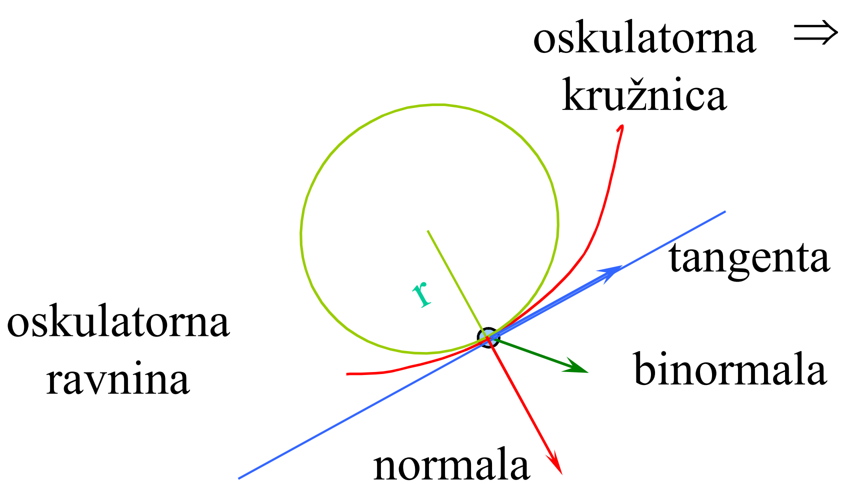
http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/apposcplane.html

6. oskulatorna ravnina u završnoj točki treba biti paralelna s a_{n-1} i a_n

$$\Rightarrow f''_{i,n}(1) = 0, \qquad i = 0\ ..\ n\text{-}2,$$
$$f''_{n-1,n}(1) \neq 0,\ \ f''_{n,n}(1) \neq 0.$$

7. simetričnost težinske funkcije - zamjena početne i završne točke povlači promjenu smjera i redoslijeda vektora.

$$\Rightarrow f_{i,n}(t) = 1 - f_{n-i+1,n}(1-t)\ , \qquad i = 1\ ..\ n.$$



$\Rightarrow$ BEZIEROVE TEŽINSKE FUNKCIJE

$$f_{i,n}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_n(t)}{d^{(i-1)}t}, \quad \Phi_n(t) = \frac{1-(1-t)^n}{-t},$$

gdje $d^{(i-1)}$ je $(i-1)$ derivacija $\qquad i = 1..n$

rekurzivni oblik pogodan za implementaciju na računalu:

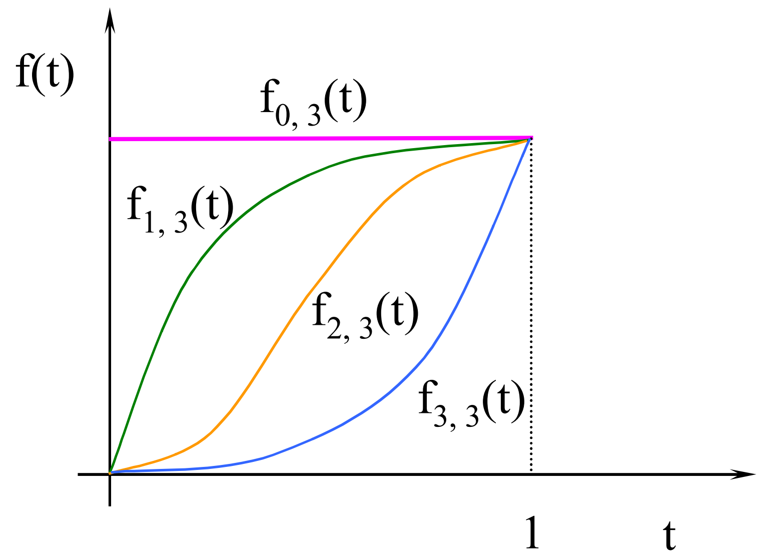
$$f_{i,n}(t) = (1-t)\, f_{i,n-1}(t) + t\, f_{i-1,n-1}(t),$$

uvjeti zaustavljanja rekurzije

$$f_{0,0}(t) = 1, \quad f_{k+1,k}(t) = 0, \quad f_{-1,k}(t) = 1.$$

* PRIMJER

Odrediti Bezierove težinske funkcije ako su zadane četiri točke.



$$\Phi_3(t) = \frac{1-(1-t)^3}{-t} = -3 + 3t - t^2,$$

$$f_{i,3}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_3(t)}{d^{(i-1)}t},$$

$$f_{0,3}(t) = 1,$$

$$f_{1,3}(t) = 3t - 3t^2 + t^3,$$

$$f_{2,3}(t) = 3t^2 - 2t^3,$$

$$f_{3,3}(t) = t^3.$$

$$\vec{p}(t) = \sum_{i=0}^{3} \vec{a}_i\, f_{i,3}(t),$$

$$\vec{p}(t) = \vec{a}_0 + \left(3t - 3t^2 + t^3\right)\vec{a}_1 + \left(3t^2 - 2t^3\right)\vec{a}_2 + t^3\vec{a}_3$$

Provjera postavljenih uvjeta na težinsku funkciju:

1. početna točka $\vec{p}(0)=\vec{a}_0$,
2. završna točka $\vec{p}(1)=\vec{a}_0 + \vec{a}_1 + \vec{a}_2 + \vec{a}_3$,

$\vec{p}'(t)=\left(3-6t+3t^2\right)\vec{a}_1+\left(6t-6t^2\right)\vec{a}_2+3t^2\vec{a}_3$,

3. derivacija u početnoj točki $\vec{p}'(0)=3\vec{a}_1$,
4. derivacija u završnoj točki $\vec{p}'(1)=3\vec{a}_3$,

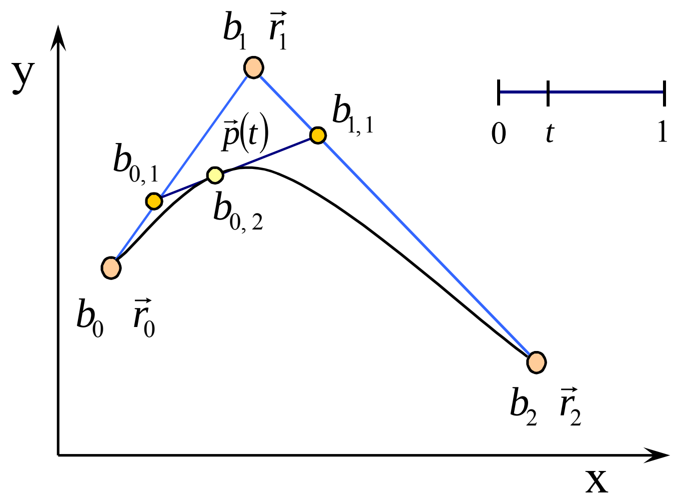
$\vec{p}''(t)=(-6+6t)\,\vec{a}_1+(6-12t)\,\vec{a}_2+6t\vec{a}_3$,

5. derivacija u početnoj točki $\vec{p}''(0)=6\left(\vec{a}_2-\vec{a}_1\right)$,
6. derivacija u završnoj točki $\vec{p}''(1)=6\left(\vec{a}_3-\vec{a}_2\right)$,

7. simetričnost $f_{1,3}(t)=1-f_{3,3}(1-t)$.

b) BERNSTEINOVE TEŽINSKE FUNKCIJE

De Casteljau - intuitivna geometrijska konstrukcija

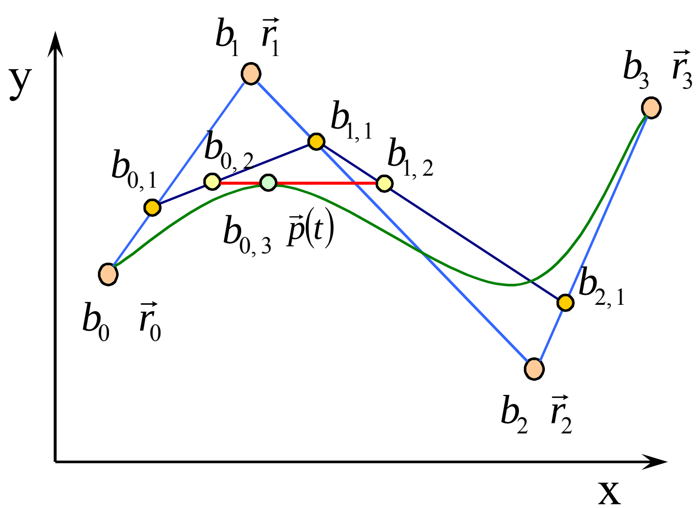


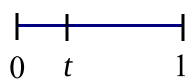
- uzastopne linearne interpolacije :

$$\left.\begin{aligned} b_{0,1} &= (1-t)b_0 + t\,b_1, \\ b_{1,1} &= (1-t)b_1 + t\,b_2, \end{aligned}\right\} \quad b_{0,2} = (1-t)b_{0,1} + t\,b_{1,1} \ ,$$

- uvrstimo :

$$b_{0,2} = (1-t)^2\, b_0 + 2t(1-t)b_1 + t^2 b_2$$





http://saltire.com/applets/spline.htm
http://www.cs.technion.ac.il/~cs234325/

- poopćenje ovog postupka daje De Casteljau-ov algoritam

$$b_{i,r} = (1-t)\, b_{i,r-1} + t\, b_{i+1,\, r-1}(t), \quad r = 1..n,\ i = 0..n-r,$$

$$b_{i,0}(t) = b_i \qquad \vec{r}_i \ \text{vrhovi kontrolnog poligona,}$$

$$b_{0,n}(t) \qquad \vec{p}(t) \ \text{točka na krivulji.}$$

$$\boxed{\vec{p}(t) = \sum_{i=0}^{n} \vec{r}_i\, b_{i,n}(t) \qquad t \in [0,\, 1]} \qquad \text{vrijedi} \qquad \sum_{i=0}^{n} b_{i,n}(t) = 1 \qquad t \in [0,\, 1]$$

$$b_{i,n}(t) \quad -\text{bazne funkcije} - \text{Bernsteinovi polinomi stupnja } n$$

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!}\, t^i (1-t)^{n-i}$$

Diskretna binomna razdioba:

t $-$ vjerojatnost događaja u svakom od $n+1$ pokušaja

$b_{i,n}$ $-$ vjerojatnost postizanja točno i događaja u $n+1$ pokušaja

* PRIMJER

Odrediti Bernsteinove težinske funkcije

ako su zadane četiri točke.

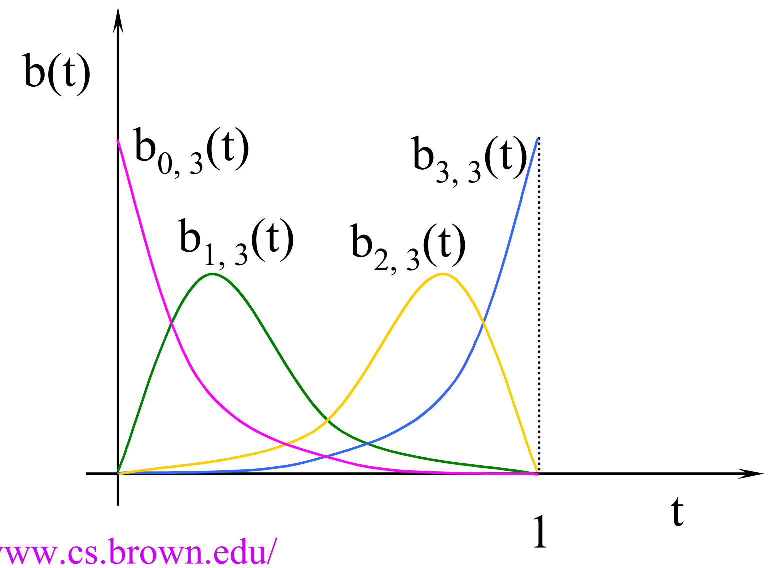
$$b_{i,3}(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i},$$

$$b_{0,3}(t) = (1-t)^3,$$

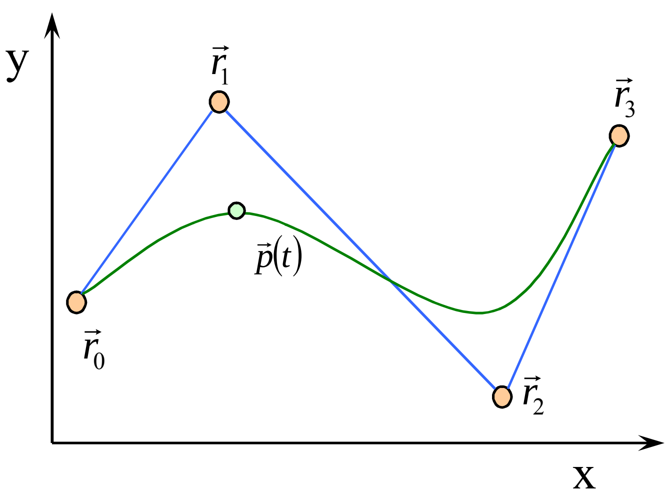
$$b_{1,3}(t) = 3t(1-t)^2,$$

$$b_{2,3}(t) = 3t^2(1-t),$$

$$b_{3,3}(t) = t^3.$$



http://www.cs.brown.edu/



$$\vec{p}(t) = \sum_{i=0}^{3} \vec{r}_i b_{i,3}(t),$$

$$\vec{p}(t) = \vec{r}_0 (1-t)^3 + 3t(1-t)^2 \vec{r}_1 + 3t^2(1-t)\vec{r}_2 + t^3 \vec{r}_3$$

$$\vec{p}'(0) = 3(\vec{r}_1 - \vec{r}_0),$$

$$\vec{p}'(1) = 3(\vec{r}_3 - \vec{r}_2).$$

Matrično:

$$\vec{p}(t)=\begin{bmatrix}t^3 & t^2 & t & t\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\vec{r}_0\\ \vec{r}_1\\ \vec{r}_2\\ \vec{r}_3\end{bmatrix}.$$

Za tangentu na Bezierovu krivulju opisanu preko Bernsteinovih težinskih funkcija vrijedi:

$\vec{p}'(0)=n(\ \vec{r}_1-\vec{r}_0),$

$\vec{p}'(1)=n(\ \vec{r}_n-\vec{r}_{n-1}).$ n..stupanj krivulje

Veza Bezierovih i Bernsteinovih težinskih funkcija:

$$f_{i,n}(t)=\sum_{j=i}^{n} b_{j,n}(t),\quad i=0..n.$$

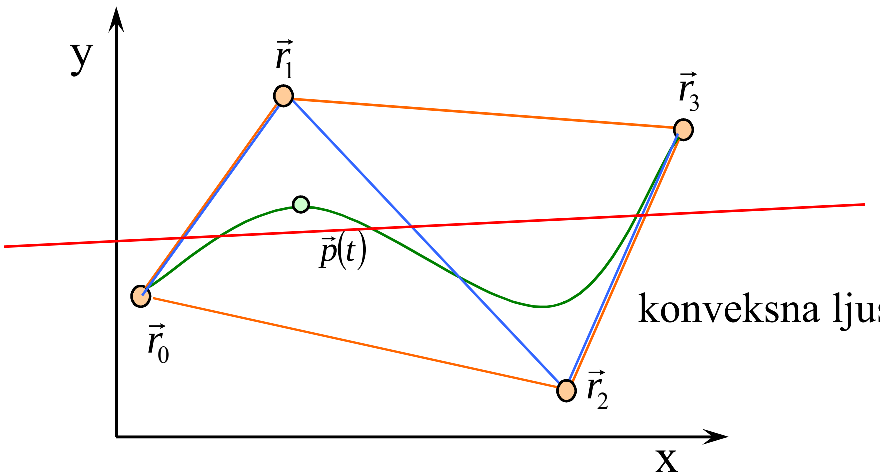
$\vec{a}_0=\vec{r}_0\,,\quad \vec{a}_i=\vec{r}_i-\vec{r}_{i-1},\qquad i=1..\ n,$

ili

$\vec{r}_0=\vec{a}_0,\quad \vec{r}_i=\vec{a}_i+\vec{r}_{i-1},\qquad i=1..\ n$

SVOJSTVA APROKSIMACIJSKIH BEZIEROVIH KRIVULJA

- postoji konveksna ljuska



suma težinskih funkcija je 1

važno kod ispitivanja sjecišta krivulje

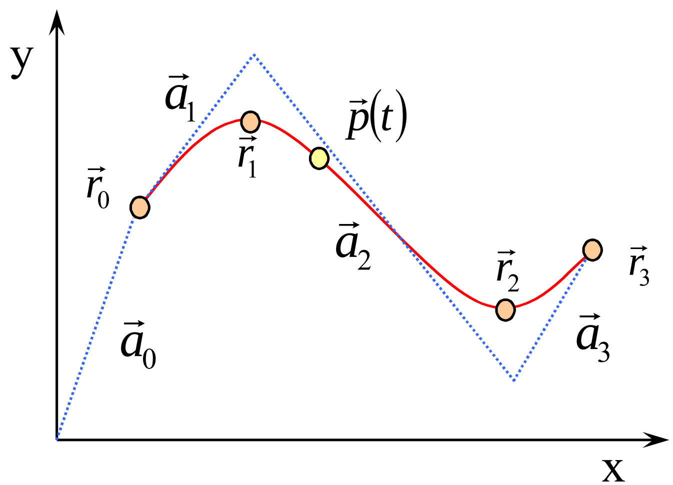
- krivulja nema više valova od kontrolnog poligona

broj sjecišta ravnine i kontrolnog poligona <= br. sjec. ravnine i krivulje

- lokalni nadzor - nije ispunjeno
- broj kontrolnih točaka je u direktnoj vezi sa stupnjem krivulje
- neovisnost o transformacijama (translacija, rotacija, skaliranje)
- simetričnost - kod uvrštenja možemo simetrično zamijeniti popis točaka
- http://www.cs.unc.edu/~mantler/research/bezier/index.html
- http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html
- http://www.ibiblio.org/e-notes/VRML/Anim/Morph.wrl

INTERPOLACIJSKE KRIVULJE BEZIERA

Prolaze svim zadanim točkama.



$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{n} \vec{a}_i f_{i,n}(t) \qquad t \in [0, 1]$$

$f_{i,n}$ – poznato na osnovi broja točaka

$\vec{a}_i$ – nepoznato – određuje se na temelju nečega poznatog ili željenog o krivulji

Potrebno je poznavati $n+1$ uvjet.

POZNATO

1. n+1 točka krivulje s vrijednošću parametra $\vec{p}_i(t_i), \quad t_i = \dfrac{i}{n}, \; i = 0..n.$

ili

2. tangente u pojedinim točkama $\vec{p}_i'(t_i) = \sum_{i=1}^{n} \vec{a}_i f_{i,n}'(t)$.

ili

3. oskulatorne ravnine, položaji centara zakrivljenosti $\vec{p}_i''(t_i) = \sum_{i=1}^{n} \vec{a}_i f_{i,n}''(t)$.

INTERPOLACIJSKA KRIVULJA KROZ n+1 TOČKU:

neka su poznate točke $\vec{p}_0 = \vec{p}(t_0), \vec{p}_1 = \vec{p}(t_1), \vec{p}_2 = \vec{p}(t_2), \ldots, \vec{p}_n = \vec{p}(t_n),$

uz parametar $t_i = \dfrac{i}{n}$ gdje $i = 0 .. n.$

$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{n} \vec{a}_i f_{i,n}(t) \qquad t \in [0, 1]$$

$$\begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \ldots \\ \vec{p}_n \end{bmatrix} = \begin{bmatrix} 1 & f_{1,n}(0) & f_{2,n}(0) & \ldots & f_{n,n}(0) \\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \ldots & f_{n,n}(t_1) \\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \ldots & f_{n,n}(t_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & f_{1,n}(1) & f_{2,n}(1) & \ldots & f_{n,n}(1) \end{bmatrix} \begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \ldots \\ \vec{a}_n \end{bmatrix}.$$

uvrstili smo: $t_0 = 0,\ t_n = 1.$

uvrstit ćemo:

za početnu točku je $\vec{p}(0)=\vec{a}_0$, tj. $f_{0,n}(0)=1,\ f_{i,n}(0)=0,\ i=1..n,$

za završnu točku je $\vec{p}(1)=\sum_{i=0}^{n}\vec{a}_i$, tj. $f_{i,n}(1)=1,\ i=0..n.$

$$\begin{bmatrix}\vec{a}_0\\ \vec{a}_1\\ \vec{a}_2\\ \dots\\ \vec{a}_n\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & \dots & 0\\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \dots & f_{n,n}(t_1)\\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \dots & f_{n,n}(t_2)\\ \dots & \dots & \dots & \dots & \dots\\ 1 & 1 & 1 & \dots & 1\end{bmatrix}^{-1}\begin{bmatrix}\vec{p}_0\\ \vec{p}_1\\ \vec{p}_2\\ \dots\\ \vec{p}_n\end{bmatrix}$$

kada odredimo nepoznate vektore $\vec{a}_i$ možemo do pojedine točke krivulje doći na osnovi Bezijer-ovih ili Bernstein-ovih težinskih funkcija.

* PRIMJER

Odrediti Interpolacijsku Bezierovu krivulju kroz četiri točke korištenjem Bezierovih težinskih funkcija.

Neka su poznate točke $\vec{p}_0 = \vec{p}(0), \vec{p}_1 = \vec{p}(1/3), \vec{p}_2 = \vec{p}(2/3), \vec{p}_3 = \vec{p}(1).$

$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{3} \vec{a}_i f_{i,3}(t) \qquad t \in [0, 1].$$

Iz prethodnog primjera za aproksimacijske Bezijerove krivulje poznate su težinske funkcije.

$$f_{0,3}(t) = 1,$$
$$f_{1,3}(t) = 3t - 3t^2 + t^3,$$
$$f_{2,3}(t) = 3t^2 - 2t^3,$$
$$f_{3,3}(t) = t^3.$$

$$\begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{19}{27} & \frac{7}{27} & \frac{1}{27} \\ 1 & \frac{26}{27} & \frac{20}{27} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}$$

$$\vec{p}(t) = \vec{a}_0 + \left(3t - 3t^2 + t^3\right)\vec{a}_1 + \left(3t^2 - 2t^3\right)\vec{a}_2 + t^3\vec{a}_3$$

4.1.2. RAZLOMLJENE FUNKCIJE

PRIKAZ KRIVULJA POMOĆU KVADRATNIH RAZLOM. FUNKCIJA

- pogodan oblik za prikaz krivulja drugog reda
- homogena koordinata omogućava prikaz koničnih krivulja

(presjek ravnine i stošca) http://www.slu.edu/classes/maymk/banchoff/CriticalPoints.html

- invarijantnost na transformaciju perspektivne projekcije (nerazlomljene krivulje su invarijantne samo na translaciju, rotaciju, skaliranje)

$$\left.\begin{aligned} x &= \frac{x_1}{x_4} = \frac{a_1 t^2 + b_1 t + c_1}{a t^2 + b t + c}, \\ y &= \frac{x_2}{x_4} = \frac{a_2 t^2 + b_2 t + c_2}{a t^2 + b t + c}, \\ z &= \frac{x_3}{x_4} = \frac{a_3 t^2 + b_3 t + c_3}{a t^2 + b t + c} \end{aligned}\right\} \text{u radnom prostoru}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \end{bmatrix} \quad \text{matrični oblik}$$

K - karakteristična matrica kvadratne krivulje, $0 \le t \le 1$,

$$\boxed{\mathbf{X} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \mathbf{K}}$$

- derivacije vektora [x_1 x_2 x_3 x_4] po parametru t - u homogenom prostoru
- matrica **K** određuje i derivacije duž krivulje

$$x_1' = \frac{d\,x_1}{d\,t} = 2a_1\,t + b_1,$$

$$x_2' = \frac{d\,x_2}{d\,t} = 2a_2\,t + b_2,$$

$$x_3' = \frac{d\,x_3}{d\,t} = 2a_3\,t + b_3,$$

$$x_4' = \frac{d\,x_4}{d\,t} = 2a\,t + b.$$

$$\mathbf{X}' = \begin{bmatrix} x_1' & x_2' & x_3' & x_4' \end{bmatrix} = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}$$

$$\mathbf{X}'' = \begin{bmatrix} x_1'' & x_2'' & x_3'' & x_4'' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}$$

$$\boxed{\mathbf{X}' = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}}$$

$$\boxed{\mathbf{X}'' = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}}$$

♣ kvadratna razlomljena krivulja određena je s tri točke

$$\mathbf{V}_0, \quad t_0 = 0, \quad \mathbf{X}_0 = \begin{bmatrix} \mathbf{V}_0 & 1 \end{bmatrix}$$

$$\mathbf{V}_1, \quad t_1 = \frac{1}{2}, \quad \mathbf{X}_1 = \begin{bmatrix} \mathbf{V}_1 & 1 \end{bmatrix}$$

$$\mathbf{V}_2, \quad t_2 = 1, \quad \mathbf{X}_2 = \begin{bmatrix} \mathbf{V}_2 & 1 \end{bmatrix}$$

tri točke uvrstimo u jednadžbu krivulje,
uzmimo da su poznati iznosi parametra

$$\mathbf{K} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix}^{-1} \mathbf{X}$$

$$\mathbf{K} = \begin{bmatrix} t_0^2 & t_0 & 1 \\ t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

* PRIMJER

Neka su zadane tri točke
i pripadni iznosi parametra.
Odrediti kvadratnu razlomljenu krivulju.

$$\mathbf{V}_0, \quad t_0 = 0, \quad \mathbf{X}_0 = \begin{bmatrix} r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_1, \quad t_1 = \frac{1}{2}, \quad \mathbf{X}_1 = \begin{bmatrix} 0 & r & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_2, \quad t_2 = 1, \quad \mathbf{X}_2 = \begin{bmatrix} -r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & 1 \\ -r & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

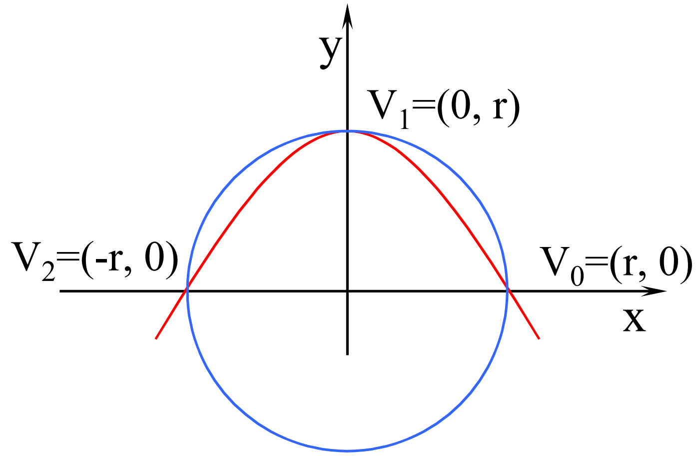
$$x_1 = -2rt + r = r(1-2t),$$

$$x_2 = -4rt^2 + 4rt = 4r(t - t^2),$$

$$x_3 = 0,$$

$$x_4 = 1.$$

po komponentama



Rezultat je parabola, to je opća krivulja drugog reda.

Ako želimo načiniti kružnicu potrebno je upotrijebiti analitičke poznate izraze za kružnicu. http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html

$$\left.\begin{array}{c} x = r\cos\varphi \\ y = r\sin\varphi \\ \hline t = \mathrm{tg}\dfrac{\varphi}{2} \end{array}\right\} \quad \left.\begin{array}{c} x = r\dfrac{1-t^2}{1+t^2} \\ y = r\dfrac{2\,t}{1+t^2} \end{array}\right\} \quad \begin{array}{c} x_1 = r\left(1-t^2\right), \\ x_2 = 2\,r\,t, \\ x_3 = 0, \\ x_4 = 1+t^2. \end{array}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

PRIKAZ KRIVULJA POMOĆU KUBNIH RAZLOMLJENIH FUNKCIJA

- kvadratnim razlomljenim funkcijama ne možemo prikazati infleksiju i ostale pojave višeg reda

$$\left.\begin{aligned}
x &= \frac{x_1}{x_4} = \frac{a_1 t^3 + b_1 t^2 + c_1 t + d_1}{a t^3 + b t^2 + ct + d}, \\
y &= \frac{x_2}{x_4} = \frac{a_2 t^3 + b_2 t^2 + c_2 t + d_2}{a t^3 + b t^2 + ct + d}, \\
z &= \frac{x_3}{x_4} = \frac{a_3 t^3 + b_3 t^2 + c_3 t + d_3}{a t^3 + b t^2 + ct + d}
\end{aligned}\right\} \text{u radnom prostoru}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \\ d_1 & d_2 & d_3 & d \end{bmatrix} \quad \text{matrični oblik}$$

A - karakteristična matrica kubne krivulje, $0 \le t \le 1$,

$$\boxed{\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}}$$

- derivacije vektora [x_1 x_2 x_3 x_4] po parametru t - u homogenom prostoru
- matrica **A** određuje i derivacije duž krivulje

$$x_1' = \frac{d\,x_1}{d\,t} = 3a_1 t^2 + 2b_1 t + c_1,$$
$$x_2' = \frac{d\,x_2}{d\,t} = 3a_2 t^2 + 2b_2 t + c_2,$$
$$x_3' = \frac{d\,x_3}{d\,t} = 3a_3 t^2 + 2b_3 t + c_3,$$
$$x_4' = \frac{d\,x_4}{d\,t} = 3a t^2 + 2b t + c.$$

$$\boxed{\mathbf{X}' = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \mathbf{A}}$$

$$\boxed{\mathbf{X}'' = \begin{bmatrix} 6t & 2 & 0 & 0 \end{bmatrix} \mathbf{A}}$$

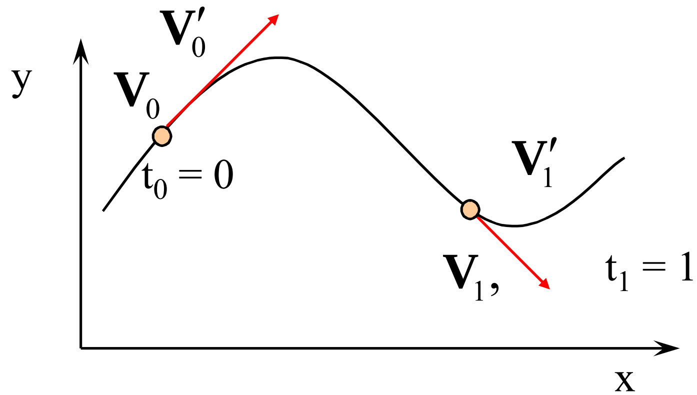
$$\boxed{\mathbf{X}''' = \begin{bmatrix} 6 & 0 & 0 & 0 \end{bmatrix} \mathbf{A}}$$

- za određivanje kubne razlomljene krivulje potrebna su četiri uvjeta

(kako bi mogli invertirati matricu)

To mogu biti 4 točke ili na primjer 2 točke i 2 derivacije.

♣ kubna razlomljena krivulja određena s dvije rubne točke i derivacije



$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \begin{bmatrix} x_4 x & x_4 y & x_4 z & x_4 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}$$

$$\mathbf{X}' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \end{pmatrix} = \begin{bmatrix} (x_4 x)' & (x_4 y)' & (x_4 z)' & x_4' \end{bmatrix} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$t_0 = 0, \quad \mathbf{X}_0 = \begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \end{bmatrix} = \begin{bmatrix} t_0^{\,3} & t_0^{\,2} & t_0 & 1 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}$$

$$t_1 = 1, \quad \mathbf{X}_1 = \begin{bmatrix} x_{41}\mathbf{V}_1 & x_{41} \end{bmatrix} = \begin{bmatrix} t_1^{\,3} & t_1^{\,2} & t_1 & 1 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{A}$$

$$t_0 = 0, \quad \mathbf{X}_0' = \begin{bmatrix} x_{40}'\mathbf{V}_0 + x_{40}\mathbf{V}_0' & x_{40}' \end{bmatrix} = \begin{bmatrix} 3t_0^{\,2} & 2t_0 & 1 & 0 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$t_1 = 1, \quad \mathbf{X}_1' = \begin{bmatrix} x_{41}'\mathbf{V}_1 + x_{41}\mathbf{V}_1' & x_{41}' \end{bmatrix} = \begin{bmatrix} 3t_1^{\,2} & 2t_1 & 1 & 0 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$\begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \\ x_{41}\mathbf{V}_1 & x_{41} \\ x'_{40}\mathbf{V}_0 + x_{40}\mathbf{V}'_0 & x'_{40} \\ x'_{41}\mathbf{V}_1 + x_{41}\mathbf{V}'_1 & x'_{41} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \\ x_{41}\mathbf{V}_1 & x_{41} \\ x'_{40}\mathbf{V}_0 + x_{40}\mathbf{V}'_0 & x'_{40} \\ x'_{41}\mathbf{V}_1 + x_{41}\mathbf{V}'_1 & x'_{41} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

$\boxed{\mathbf{A} = \mathbf{MHV}}$

M.........univerzalna transformacijska matrica

- Segment krivulje određen rubnim točkama i derivacijama u njima

$$\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \mathbf{MHV}$$

M - ne ovisi o obliku krivulje već o izboru točaka (derivacija)

H - krivulja prolazi početnom i krajnjom točkom uz zadane derivacije, a derivacija homogene komponente određuje kako će prolaziti

- ako je $x'_{40} = x'_{41} = 0$ dobit ćemo specijalan slučaj odnosno običnu parametarsku kubnu krivulju koja se zove HERMITOVA KRIVULJA

V - zadane točke i derivacije koje određuju segment krivulje u radnom prostoru

$$\mathbf{M} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

- HERMITOVA KRIVULJA

$$\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}_0' & 0 \\ \mathbf{V}_1' & 0 \end{bmatrix}$$

- VEZA HERMITOVE I BEZIEROVE KRIVULJE (preko Bernsteinovih polinoma)

$$\vec{p}(t) = \begin{bmatrix} t^3 & t^2 & t & t \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_0 \\ \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}. \qquad \begin{aligned} \mathbf{V}_0 &= \vec{r}_0 \\ \mathbf{V}_1 &= \vec{r}_3, \\ \mathbf{V}_0' &= 3(\vec{r}_1 - \vec{r}_0) \\ \mathbf{V}_1' &= 3(\vec{r}_3 - \vec{r}_2) \end{aligned}$$

$\Rightarrow$ radi se o istoj krivulji

* PRIMJER

Neka su zadane dvije točke
i derivacije u njima.
Odrediti kubnu razlomljenu krivulju.

$$\mathbf{V}_0 = [0 \quad 0 \quad 0], \quad t_0 = 0,$$
$$\mathbf{V}_1 = [1 \quad 0 \quad 0], \quad t_1 = 1,$$
$$\mathbf{V}_0' = [1 \quad 1 \quad 0], \quad t_0 = 0,$$
$$\mathbf{V}_1' = [1 \quad -1 \quad 0], \quad t_1 = 1.$$

$$[x_{40} \quad x_{41} \quad x_{40}' \quad x_{41}'] = [1 \quad 1 \quad a \quad b]$$

$$\mathbf{A} = \mathbf{MHV} =$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 & 0 & a+b \\ -b & -1 & 0 & -(2a+b) \\ 1 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad x_4] = [t^3 \quad t^2 \quad t \quad 1]\,\mathbf{A}$$

$$\left.\begin{aligned} x &= \frac{x_1}{x_4} = \frac{bt^3 - bt^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ y &= \frac{x_2}{x_4} = \frac{-t^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ z &= \frac{x_3}{x_4} = 0 \end{aligned}\right\} \quad a,\, b = ?$$

Uvodimo dodatnu točku $V_2=(1/2\ 1/2\ 0)$, $t_2=1/2$. $\Rightarrow$ a=-2, b=2

http://www.rose-hulman.edu/~finn/courses/MA323GeomModel/TestApplets/RationalC2Spline.html

$$x = \frac{2t^3 - 2t^2 + t}{2t^2 - 2t + 1},$$

$$y = \frac{-t^2 + t}{2t^2 - 2t + 1},$$

$$z = 0$$

VEZA KOORDINATA I PARAMETARSKIH DERIVACIJA IZMEĐU RADNOG I HOMOGENOG PROSTORA

- radni prostor:

$$\mathrm{V}(t) = [\mathrm{x}(t) \quad \mathrm{y}(t) \quad \mathrm{z}(t)]. \qquad \frac{d\,\mathrm{V}(t)}{dt} = \begin{bmatrix} \frac{\mathrm{x}(t)}{dt} & \frac{\mathrm{y}(t)}{dt} & \frac{\mathrm{z}(t)}{dt} \end{bmatrix}.$$

- homogeni prostor

$$\mathbf{X}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]. \qquad \frac{d\,\mathbf{X}(t)}{dt} = \begin{bmatrix} \frac{\mathrm{x}_1(t)}{dt} & \frac{\mathrm{x}_2(t)}{dt} & \frac{\mathrm{x}_3(t)}{dt} & \frac{\mathrm{x}_4(t)}{dt} \end{bmatrix}.$$

- VEZA KOORDINATA

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad y = \frac{x_3}{x_4} \qquad x_1 = x\,x_4, \quad x_2 = y\,x_4, \quad x_3 = y\,x_4$$

$$\mathbf{X} = (x_1 \quad x_2 \quad x_3 \quad x_4) = [x_4 x \quad x_4 y \quad x_4 z \quad x_4] = x_4[x \quad y \quad z \quad 1] = x_4[\mathbf{V} \quad 1]$$

$$\boxed{\mathbf{X} = x_4[\mathbf{V} \quad 1]}$$

- VEZA PRVE DERIVACIJE - homogena komponenta nije konstanta

$$\mathbf{X}' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \end{pmatrix} = \begin{bmatrix} (x_4 x)' & (x_4 y)' & (x_4 z)' & x_4' \end{bmatrix} =$$

$$\left[(x_4' x + x_4 x') \quad (x_4' y + x_4 y') \quad (x_4' z + x_4 z') \quad x_4' \right] = \begin{bmatrix} x_4' & x_4 \end{bmatrix} \begin{bmatrix} x & y & z & 1 \\ x' & y' & z' & 0 \end{bmatrix}$$

$$\boxed{\mathbf{X}' = \begin{bmatrix} x_4' & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix}}$$

- VEZA DRUGE DERIVACIJE

$$\mathbf{X}'' = (\mathbf{X}')' = \left(\begin{bmatrix} x_4' & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix} \right)' = \left((x_4' \mathbf{V} + x_4 \mathbf{V}') \quad x_4' \right)' = \begin{bmatrix} (x_4 \mathbf{V})'' & x_4'' \end{bmatrix} =$$

$$= \left[\left(x_4'' \mathbf{V} + 2\, x_4' \mathbf{V}' + x_4 \mathbf{V}'' \right) \quad x_4'' \right]$$

$$\boxed{\mathbf{X}'' = \begin{bmatrix} x_4'' & 2x_4' & x_4 \end{bmatrix} \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \\ \mathbf{V}'' & 0 \end{bmatrix}}$$

* PRIMJER

Odrediti prvu derivaciju u homogenom i radnom prostoru na kružnicu.

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

$\underline{\text{homogeni prostor}}:$

$$x_1 = r\left(1-t^2\right),$$
$$x_2 = 2rt,$$
$$x_4 = t^2 + 1.$$

$$x_1' = -2rt,$$
$$x_2' = 2r,$$
$$x_4' = 2t.$$

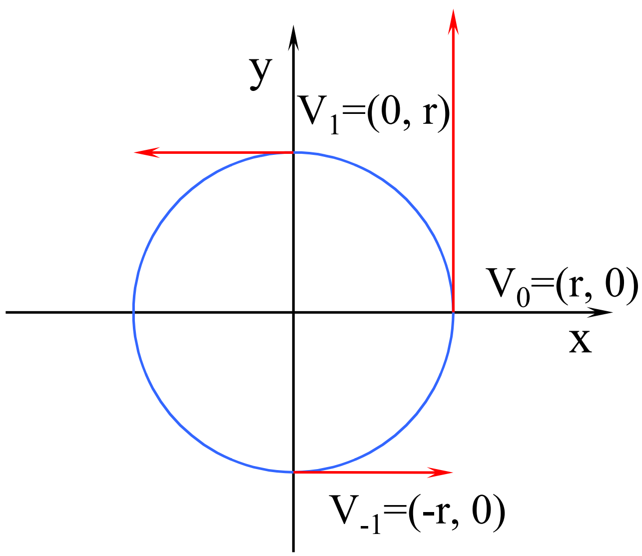
$\underline{\text{radni prostor}}:$

$$x = \frac{r\left(1-t^2\right)}{1+t^2},$$
$$y = \frac{2rt}{1+t^2}.$$

$$x' = \left(\frac{x_1}{x_4}\right)' = \frac{-4rt}{1+2t^2+t^4} \neq \frac{x_1'}{x_4'},$$

$$y' = \left(\frac{x_2}{x_4}\right)' = \frac{2r\left(1-t^2\right)}{1+2t^2+t^4} \neq \frac{x_2'}{x_4'}.$$

t	x_1	x_2	x_4	x	y	x'_1	x'_2	x'_4	x'	y'
0	r	0	1	r	0	0	2r	0	0	2r
1	0	2r	2	0	r	-2r	2r	2	-r	0
-1	0	-2r	2	0	-r	2r	2r	-2	r	0



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ nagib tangente

http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/speed.html