5.2 Stochastic Process

- Let the random variable X_t denote the value of an economic factor (e.g. stock price, interest rate etc.) at time t.
- If $\{x_1, x_2, \ldots, x_n\}$ is an observed data from time 1 up to time n, then $\{x_1, x_2, \ldots, x_n\}$ is a time series for the relevant economic factor. Furthermore, such empirical data can be thought of a particular realization of a stochastic process.
- In general, such stochastic process can be described by an n-dimensional probability distribution $p(x_1, x_2, \ldots, x_n)$.
- Assuming joint normality, such a distribution is described by n means $\mathrm{E}(x_1), \mathrm{E}(x_2), \ldots, \mathrm{E}(x_n)$; n variances $\mathrm{Var}(x_1), \mathrm{Var}(x_2), \ldots, \mathrm{Var}(x_n)$; and $n(n-1)/2$ covariances $\mathrm{Cov}(x_i, x_j)$, $i < j$.
- The special case $n = 3$ is plotted in Figure 5.1; the distribution is described by nine parameters (three means, three variances, three covariances).

Figure 5.1: Probability Distributions for a General Stochastic Process ($n = 3$)

- To infer such a general probability structure from just one realization of the stochastic process will be impossible, since there are n observations but $n+n+n(n-1)/2$ unknown parameters.
- Hence some simplifying assumptions have to be made $\Rightarrow$ *stationarity.*

5.2.1 Stationary

Definition 5.2.1 *The process $X = \{x_t : t \geq 0\}$, taking values in $\Re$, is called **strongly stationary** if the joint probability distribution of a set of m observations at times $t_1, t_2, \ldots, t_m$ is identical to the joint probability distribution of the observations at times $t_1 + k, t_2 + k, \ldots, t_m + k$, for any k.*

- For instance, if $m = 1$, this implies that the marginal distribution at time t is the same as the marginal distribution at any other point in time; $p(x_t) = p(x_{t+k})$; i.e. the marginal distribution does not depend on time, which in turn implies that the mean $\mathrm{E}(x_t) = \mu$, and the variance $\mathrm{Var}(x_t) = \gamma_0$ are constant. (see Figure 5.2)

Figure 5.2: Probability Distributions for a Stationary Stochastic Process ($n = 3$)

- If $m = 2$, stationary implies that all bivariate distributions $p(x_t, x_{t-k})$ do not depend on t; thus the covariances $\mathrm{Cov}(x_t, x_{t-k})$ are only functions of the lag k, but not of time t (i.e. $\mathrm{Cov}(x_1, x_{1+k}) = \mathrm{Cov}(x_2, x_{2+k}) = \cdots = \mathrm{Cov}(x_{n-k}, x_n)$, for all k).

- The stationarity condition implies that the mean and variance of the process are constant and that the *autocovariances*

$$\gamma_k = \mathrm{Cov}(x_t, x_{t-k}) = \mathrm{E}[(x_t - \mu)(x_{t-k} - \mu)] \tag{5.2.1}$$

  and the *autocorrelations*

$$\rho_k = \frac{\mathrm{Cov}(x_t, x_{t-k})}{[\mathrm{Var}(x_t) \cdot \mathrm{Var}(x_{t-k})]^{1/2}} = \frac{\gamma_k}{\gamma_0} \tag{5.2.2}$$

  depend only on the lag (or time difference) k.

- Since these conditions apply only to the first- and second-order moments of the process, it is also called *second-order* or *weak stationarity.*

- If a series a weakly stationary and normally distributed, then it must be stationary in the strong sense.

- Note that $\gamma_k = \gamma_{-k}$ and $\rho_k = \rho_{-k}$.

- **Sample Autocorrelation Function:**

$$\hat{\rho}_k = \frac{\sum_{t=k+1}(x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}, \qquad k = 0, 1, 2, \ldots$$

- For every weakly stationary nondeterministic stochastic process, $(x_t - \mu)$, it can be written as a linear combination (or linear filter) of a sequence of uncorrelated random variables. The linear representation is given by

$$x_t - \mu = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \tag{5.2.3}$$

with $\psi_0 = 1$.

  - The random variables $\{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ are a sequence of uncorrelated r.v. from a fixed distribution with

$$\begin{aligned} \mathrm{E}(\varepsilon_t) &= 0 && (5.2.4)\\ \mathrm{Var}(\varepsilon_t) &= \mathrm{E}(\varepsilon_t^2) = \sigma^2 && (5.2.5)\\ \mathrm{E}(\varepsilon_s \varepsilon_t) &= 0, \qquad s \neq t. && (5.2.6) \end{aligned}$$

    Such a sequence is usually referred to as a *white noise process.*
  - These r.v. define the *shocks* to the system.
  - If in addition to Conditions (5.2.4)-(5.2.6), ε_s and ε_t, $s \neq t$, are independent and that

$$\varepsilon_t \sim N(0, \sigma^2),$$

    we have the *Gaussian white noise process.*
  - The ψ_j weights in (5.2.3) are the coefficients in this linear combination; their number can be either finite or infinite.

$$\begin{aligned} \mathrm{E}(x_t) &= \mu \\ \mathrm{Var}(x_t) &= \gamma_0 = \mathrm{E}[(x_t - \mu)^2] \\ &= \\ &= \\ &= \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \\ \mathrm{Cov}(x_t, x_{t+k}) &= \gamma_k = \mathrm{E}[(x_t - \mu)(x_{t+k} - \mu)] \\ &= \mathrm{E}[(\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots) \\ &\qquad \times (\varepsilon_{t+k} + \psi_1 \varepsilon_{t+k-1} + \cdots + \psi_k \varepsilon_t + \psi_{k+1} \varepsilon_{t-1} + \cdots)] \\ &= \sigma^2 (\psi_k + \psi_1 \psi_{k+1} + \psi_2 \psi_{k+2} + \cdots) \\ &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k} \end{aligned}$$

    since $\mathrm{E}(\varepsilon_{t-i} \varepsilon_{t-j}) = 0$ for $i \neq j$.

$$\rho_k = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+k}}{\sum_{j=0}^{\infty} \psi_j^2}.$$

– If the coefficient ψ_j is infinite, then some assumptions concerning the convergence of these coefficients are needed. In fact, we have to assume that the weights converge absolutely ($\sum_{j=0}^{\infty} |\psi_j| < \infty$). This condition, which is equivalent to the stationarity assumption, guarantees that all moment exist and are independent of time t.

5.2.2 Moving Average Process

First-order Moving Average Process: MA(1)

Letting $\psi_1 = -\theta$ and $\psi_j = 0, j > 1$, the model (5.2.3) leads to

$$x_t - \mu = \varepsilon_t - \theta\varepsilon_{t-1}.$$

This time-series is called a *first-order moving average process*, denoted by $MA(1)$.

$$\begin{aligned} \mathrm{E}[x_t] &= \mu \\ \gamma_0 = \mathrm{Var}[x_t] &= \mathrm{E}(\varepsilon_t - \theta\varepsilon_{t-1})^2 \\ &= \mathrm{E}(\varepsilon_t^2 - 2\theta\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2) \\ &= \sigma^2 + 0 + \theta^2\sigma^2 \\ &= (1+\theta^2)\sigma^2 \end{aligned}$$

The first autocovariance:

$$\begin{aligned} \gamma_1 &= \mathrm{E}[(x_t - \mu)(x_{t-1} - \mu)] \\ &= \mathrm{E}[(\varepsilon_t - \theta\varepsilon_{t-1})(\varepsilon_{t-1} - \theta\varepsilon_{t-2})] \\ &= \\ &= \\ &= -\theta\sigma^2 \end{aligned}$$

Higher autocovariance: $\gamma_k = 0, k > 1$, so that the autocorrelation function:

$$\rho_1 = \frac{-\theta}{1+\theta^2}, \qquad \rho_k = 0, \quad for \quad k > 1.$$

This implies that observation one step apart are correlated. However, observations more than one step apart are uncorrelated. (see Figure 5.3)

Moving Average Process of Order q: $MA(q)$

Defined by

$$x_t - \mu = \varepsilon_t - \theta_1\varepsilon_{t-1} - \cdots - \theta_q\varepsilon_{t-q}; \tag{5.2.7}$$

i.e. $MA(q)$ is obtained by setting

$$\psi_0 = 1, \quad \psi_j = -\theta_j, \quad for\ j = 1, \ldots q, \qquad \psi_j = 0, \quad for\ j > q$$

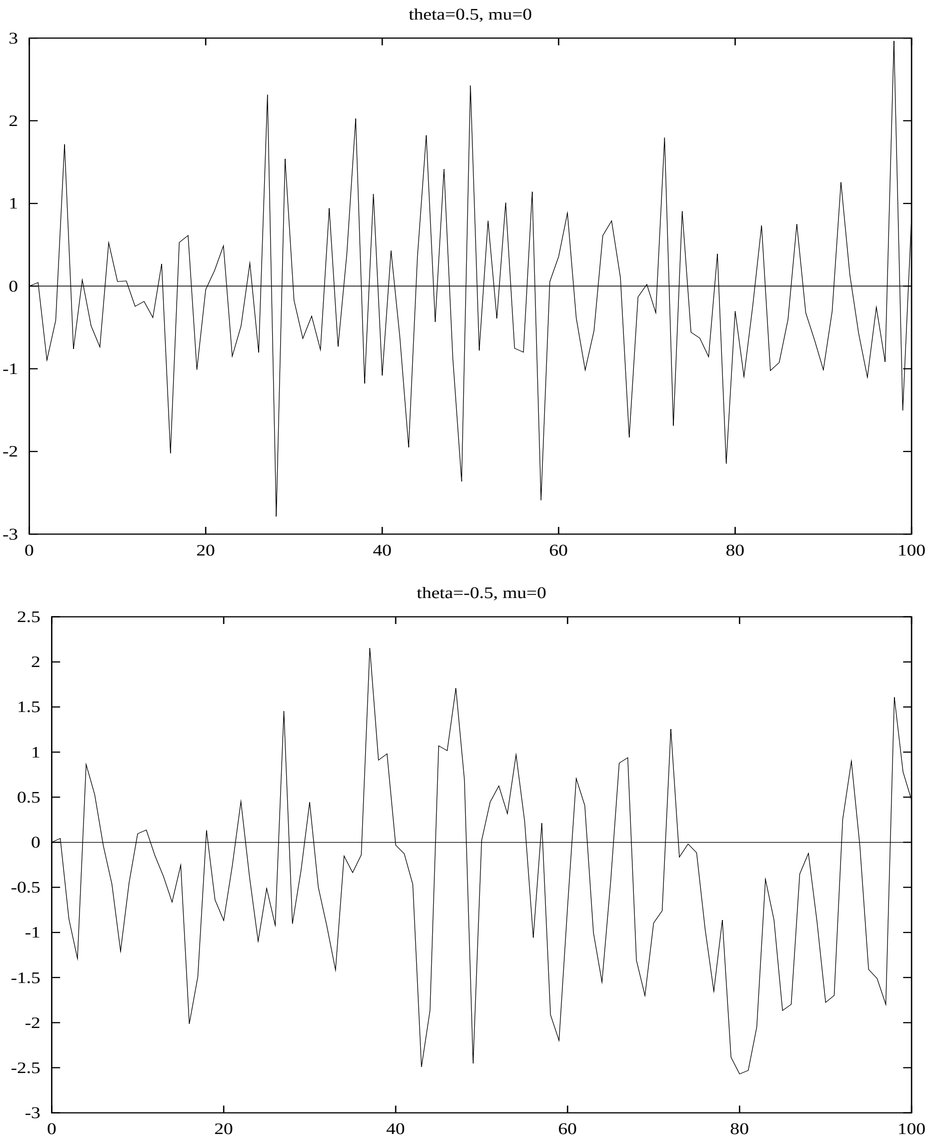

Figure 5.3: Realizations of MA(1) process for $\theta = 0.5, -0.5$

$$
\begin{aligned}
\gamma_0 &= (1+\theta_1^2+\cdots+\theta_q^2)\sigma^2 \\
\gamma_k &= (-\theta_k+\theta_1\theta_{k+1}+\cdots+\theta_{q-k}\theta_q)\sigma^2, \quad k=1,\ldots,q \\
\gamma_k &= 0, \quad k>q
\end{aligned}
$$

5.2.3 Autoregressive Process

First-order Autoregressive Process [AR(1)]

- The choice of $\psi_j = \phi^j$ in (5.2.3) leads to the process of the form

$$
\begin{aligned}
x_t - \mu &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots \\
&= \varepsilon_t + \phi(\varepsilon_{t-1} + \phi\varepsilon_{t-2} + \phi^2\varepsilon_{t-3} + \cdots) \\
&= \phi(x_{t-1} - \mu) + \varepsilon_t
\end{aligned}
$$

  The above process is known as *first-order autoregressive process*, $AR(1)$.

- In this AR(1) process, we must ensure $|\phi| < 1$; otherwise $\sum_{j=0}^{\infty} |\psi_j|$ would not converge. Furthermore, we have

$$
\begin{aligned}
\gamma_0 &= \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2}{1-\phi^2} \\
\gamma_k &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k} \\
&= \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+k} \\
&= \frac{\sigma^2 \phi^k}{1-\phi^2} = \phi^k \gamma_0, \qquad k = 0, 1, 2, \ \ldots \\
\rho_k &= \phi^k
\end{aligned}
$$

  Note that the autocorrelations decay geometrically to zero, and for $\phi < 0$ the autocorrelations decay in an oscillatory pattern. (see Figure 5.4)

Autoregressive Process of Order p [AR(p)]

$$
x_t - \mu = \varepsilon_t + \phi_1(x_{t-1} - \mu) + \cdots + \phi_p(x_{t-p} - \mu)
$$

5.2.4 Autoregressive Moving Average Process [ARMA(p,q)]

It is possible to have a process which combines both AR(p) and MA(q). This results in so-called $ARMA(p,q)$:

$$
x_t - \mu = \phi_1(x_{t-1} - \mu) + \cdots + \phi_p(x_{t-p} - \mu) + \varepsilon_t - \theta_1\varepsilon_{t-1} - \cdots - \theta_q\varepsilon_{t-q} \tag{5.2.8}
$$

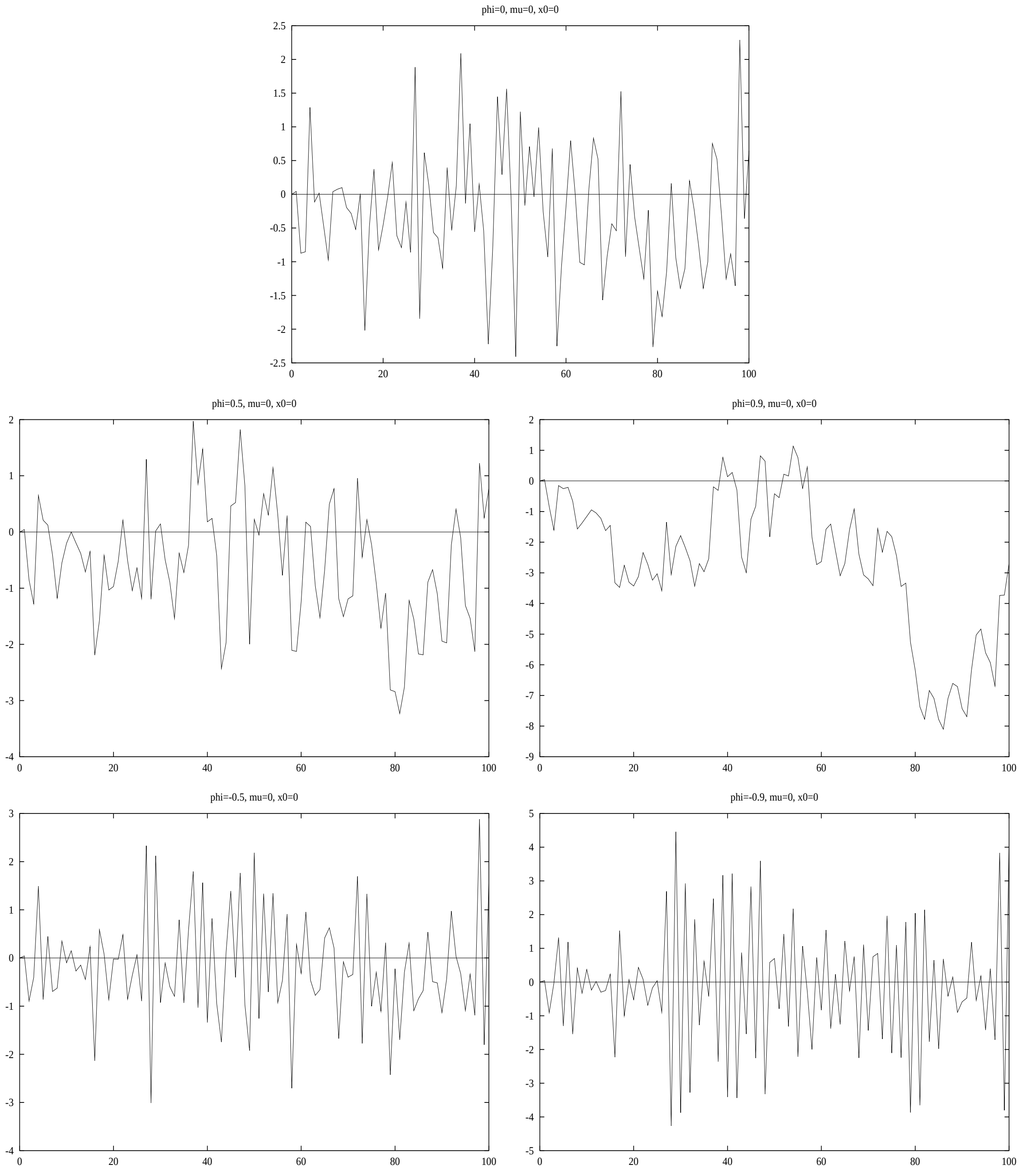

Figure 5.4: Realizations of AR(1) process for various values of ϕ